CHAPTER ONE

Solutions for Section 1.1

Exercises

1. Since t represents the number of years since 2010, we see that $f(5)$ represents the population of the city in 2015. In 2015, the city's population was 7 million.

2. Since $T = f(P)$, we see that $f(200)$ is the value of T when $P = 200$; that is, the thickness of pelican eggs when the concentration of PCBs is 200 ppm.

3. If there are no workers, there is no productivity, so the graph goes through the origin. At first, as the number of workers increases, productivity also increases. As a result, the curve goes up initially. At a certain point the curve reaches its highest level, after which it goes downward; in other words, as the number of workers increases beyond that point, productivity decreases. This might, for example, be due either to the inefficiency inherent in large organizations or simply to workers getting in each other's way as too many are crammed on the same line. Many other reasons are possible.

4. The slope is $(1-0)/(1-0) = 1$. So the equation of the line is $y = x$.

5. The slope is $(3-2)/(2-0) = 1/2$. So the equation of the line is $y = (1/2)x + 2$.

6. The slope is

$$\text{Slope } = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}.$$

Now we know that $y = (1/2)x + b$. Using the point $(-2, 1)$, we have $1 = -2/2 + b$, which yields $b = 2$. Thus, the equation of the line is $y = (1/2)x + 2$.

7. The slope is $\dfrac{6-0}{2-(-1)} = 2$ so the equation of the line is $y - 6 = 2(x-2)$ or $y = 2x + 2$.

8. Rewriting the equation as $y = -\dfrac{5}{2}x + 4$ shows that the slope is $-\dfrac{5}{2}$ and the vertical intercept is 4.

9. Rewriting the equation as

$$y = -\frac{12}{7}x + \frac{2}{7}$$

shows that the line has slope $-12/7$ and vertical intercept $2/7$.

10. Rewriting the equation of the line as

$$-y = \frac{-2}{4}x - 2$$
$$y = \frac{1}{2}x + 2,$$

we see the line has slope $1/2$ and vertical intercept 2.

11. Rewriting the equation of the line as

$$y = \frac{12}{6}x - \frac{4}{6}$$
$$y = 2x - \frac{2}{3},$$

we see that the line has slope 2 and vertical intercept $-2/3$.

12. (a) is (V), because slope is positive, vertical intercept is negative
(b) is (IV), because slope is negative, vertical intercept is positive
(c) is (I), because slope is 0, vertical intercept is positive
(d) is (VI), because slope and vertical intercept are both negative
(e) is (II), because slope and vertical intercept are both positive
(f) is (III), because slope is positive, vertical intercept is 0

13. (a) is (V), because slope is negative, vertical intercept is 0
(b) is (VI), because slope and vertical intercept are both positive
(c) is (I), because slope is negative, vertical intercept is positive
(d) is (IV), because slope is positive, vertical intercept is negative
(e) is (III), because slope and vertical intercept are both negative
(f) is (II), because slope is positive, vertical intercept is 0

14. The intercepts appear to be $(0,3)$ and $(7.5,0)$, giving

$$\text{Slope } = \frac{-3}{7.5} = -\frac{6}{15} = -\frac{2}{5}.$$

The y-intercept is at $(0,3)$, so a possible equation for the line is

$$y = -\frac{2}{5}x + 3.$$

(Answers may vary.)

15. $y - c = m(x - a)$

16. Given that the function is linear, choose any two points, for example $(5.2, 27.8)$ and $(5.3, 29.2)$. Then

$$\text{Slope} = \frac{29.2 - 27.8}{5.3 - 5.2} = \frac{1.4}{0.1} = 14.$$

Using the point-slope formula, with the point $(5.2, 27.8)$, we get the equation

$$y - 27.8 = 14(x - 5.2)$$

which is equivalent to

$$y = 14x - 45.$$

17. $y = 5x - 3$. Since the slope of this line is 5, we want a line with slope $-\frac{1}{5}$ passing through the point $(2,1)$. The equation is $(y-1) = -\frac{1}{5}(x-2)$, or $y = -\frac{1}{5}x + \frac{7}{5}$.

18. The line $y+4x = 7$ has slope -4. Therefore the parallel line has slope -4 and equation $y-5 = -4(x-1)$ or $y = -4x+9$. The perpendicular line has slope $\frac{-1}{(-4)} = \frac{1}{4}$ and equation $y - 5 = \frac{1}{4}(x-1)$ or $y = 0.25x + 4.75$.

19. The line parallel to $y = mx + c$ also has slope m, so its equation is

$$y = m(x - a) + b.$$

The line perpendicular to $y = mx + c$ has slope $-1/m$, so its equation will be

$$y = -\frac{1}{m}(x - a) + b.$$

20. Since the function goes from $x = 0$ to $x = 4$ and between $y = 0$ and $y = 2$, the domain is $0 \le x \le 4$ and the range is $0 \le y \le 2$.

21. Since x goes from 1 to 5 and y goes from 1 to 6, the domain is $1 \le x \le 5$ and the range is $1 \le y \le 6$.

22. Since the function goes from $x = -2$ to $x = 2$ and from $y = -2$ to $y = 2$, the domain is $-2 \le x \le 2$ and the range is $-2 \le y \le 2$.

23. Since the function goes from $x = 0$ to $x = 5$ and between $y = 0$ and $y = 4$, the domain is $0 \le x \le 5$ and the range is $0 \le y \le 4$.

24. The domain is all numbers. The range is all numbers ≥ 2, since $x^2 \ge 0$ for all x.

25. The domain is all x-values, as the denominator is never zero. The range is $0 < y \le \dfrac{1}{2}$.

26. The value of $f(t)$ is real provided $t^2 - 16 \ge 0$ or $t^2 \ge 16$. This occurs when either $t \ge 4$, or $t \le -4$. Solving $f(t) = 3$, we have

$$\begin{aligned}\sqrt{t^2 - 16} &= 3\\ t^2 - 16 &= 9\\ t^2 &= 25\end{aligned}$$

so

$$t = \pm 5.$$

27. We have $V = kr^3$. You may know that $V = \frac{4}{3}\pi r^3$.

28. If distance is d, then $v = \frac{d}{t}$.

29. For some constant k, we have $S = kh^2$.

30. We know that E is proportional to v^3, so $E = kv^3$, for some constant k.

31. We know that N is proportional to $1/l^2$, so

$$N = \frac{k}{l^2}, \quad \text{for some constant } k.$$

Problems

32. (a) Each date, t, has a unique daily snowfall, S, associated with it. So snowfall is a function of date.
(b) On December 12, the snowfall was approximately 5 inches.
(c) On December 11, the snowfall was above 10 inches.
(d) Looking at the graph we see that the largest increase in the snowfall was between December 10 to December 11.

33. (a) When the car is 5 years old, it is worth \$6000.
(b) Since the value of the car decreases as the car gets older, this is a decreasing function. A possible graph is in Figure 1.1:

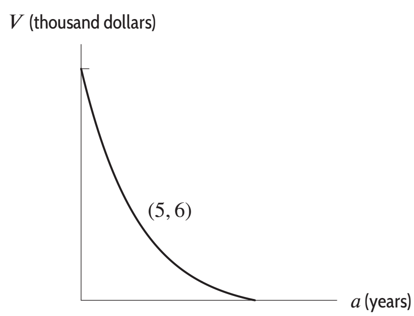


Figure 1.1

(c) The vertical intercept is the value of V when $a = 0$, or the value of the car when it is new. The horizontal intercept is the value of a when $V = 0$, or the age of the car when it is worth nothing.

34. (a) The story in (a) matches Graph (IV), in which the person forgot her books and had to return home.
(b) The story in (b) matches Graph (II), the flat tire story. Note the long period of time during which the distance from home did not change (the horizontal part).
(c) The story in (c) matches Graph (III), in which the person started calmly but sped up later.

The first graph (I) does not match any of the given stories. In this picture, the person keeps going away from home, but his speed decreases as time passes. So a story for this might be: *I started walking to school at a good pace, but since I stayed up all night studying calculus, I got more and more tired the farther I walked.*

35. The year 2000 was 12 years before 2012 so 2000 corresponds to $t = 12$. Thus, an expression that represents the statement is:

$$f(12) = 7.049.$$

36. The year 2012 was 0 years before 2012 so 2012 corresponds to $t = 0$. Thus, an expression that represents the statement is:

$$f(0) \text{ meters}.$$

37. The year 1949 was $2012 - 1949 = 63$ years before 2012 so 1949 corresponds to $t = 63$. Similarly, we see that the year 2000 corresponds to $t = 12$. Thus, an expression that represents the statement is:

$$f(63) = f(12).$$

38. Since $t = 1$ means one year before 2012, then $t = 1$ corresponds to the year 2011. Similarly, $t = 0$ corresponds to the year 2012. Thus, $f(1)$ and $f(0)$ are the average annual sea level values, in meters, in 2011 and 2012, respectively. Because 8 millimeters is the same as 0.008 meters, the average sea level in 2012, $f(0)$, is 0.008 less than the sea level in 2011 which is $f(1)$. An expression that represents the statement is:

$$f(0) = f(1) - 0.008.$$

Note that there are other possible equivalent expressions, such as: $f(0) - f(1) = -0.008$.

39. See Figure 1.2.

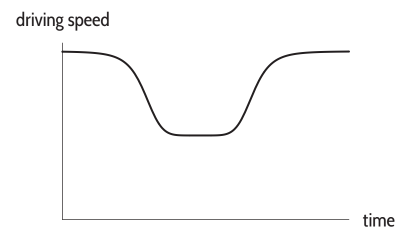


Figure 1.2

40. See Figure 1.3.

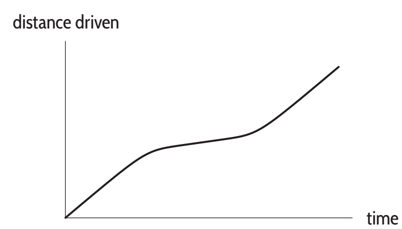


Figure 1.3

41. See Figure 1.4.

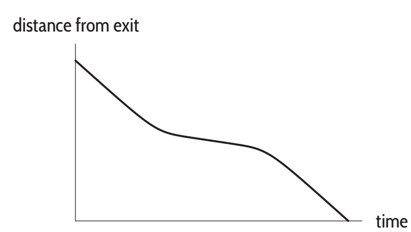


Figure 1.4

42. See Figure 1.5.

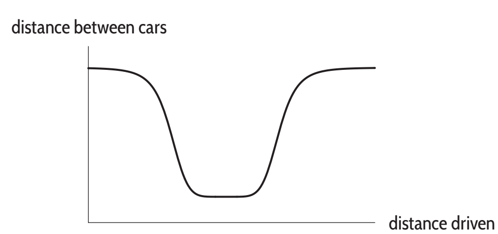


Figure 1.5

43. (a) $f(30) = 10$ means that the value of f at $t = 30$ was 10. In other words, the temperature at time $t = 30$ minutes was 10°C. So, 30 minutes after the object was placed outside, it had cooled to 10 °C.

(b) The intercept a measures the value of $f(t)$ when $t = 0$. In other words, when the object was initially put outside, it had a temperature of a°C. The intercept b measures the value of t when $f(t) = 0$. In other words, at time b the object's temperature is 0 °C.

44. (a) The height of the rock decreases as time passes, so the graph falls as you move from left to right. One possibility is shown in Figure 1.6.

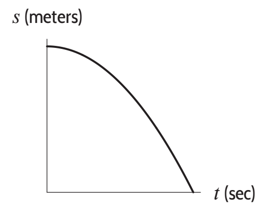


Figure 1.6

(b) The statement $f(7) = 12$ tells us that 7 seconds after the rock is dropped, it is 12 meters above the ground.

(c) The vertical intercept is the value of s when $t = 0$; that is, the height from which the rock is dropped. The horizontal intercept is the value of t when $s = 0$; that is, the time it takes for the rock to hit the ground.

45. See Figure 1.7.

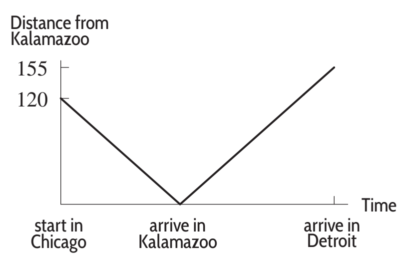


Figure 1.7

46. (a) We select two points on the line. Using the values (1992, 6.75) and (2006, 12.25) we have

$$\text{Slope} = \frac{12.25 - 6.75}{2006 - 1992 \text{ years}} = 0.4 \, \frac{\text{million barrels per day}}{\text{year}}.$$

(b) With t as the year and $f(t)$ as the quantity of imports in millions of barrels per day, we have

$$f(t) = 6.75 + 0.4(t - 1992).$$

Alternatively, if t as the number of years since 1992 and if $f(t)$ is the quantity of imports in millions of barrels per day

$$f(t) = 6.75 + 0.4t.$$

(c) We have

$$\begin{aligned} 6.75 + 0.4t &= 18 \\ t &= 28.125. \end{aligned}$$

The model predicts imports will reach 18 million barrels a day in the year $1992 + 28.125 = 2020.125$.

This prediction could serve as a guideline, but it is very risky to put too much reliance on a prediction more than ten years into the future. Many unexpected events could drastically change the economic environment during that time. This prediction should be used with caution.

47. (a) Reading coordinates from the graph, we see that rainfall $r = 100$ mm corresponds to about $Q = 600$ kg/hectare, and $r = 600$ mm corresponds to about $Q = 5800$ kg/hectare. Using a difference quotient we have

$$\text{Slope} = \frac{\Delta Q}{\Delta r} = \frac{5800 - 600}{600 - 100} = 10.4 \text{ kg/hectare per mm.}$$

(b) Every additional 1 mm of annual rainfall corresponds to an additional 10.4 kg of grass per hectare.

(c) Using the slope, we see that the equation has the form

$$Q = b + 10.4r.$$

Substituting $r = 100$ and $Q = 600$ we can solve for b.

$$\begin{aligned} b + 10.4(100) &= 600 \\ b &= -440. \end{aligned}$$

The equation of the line is

$$Q = -440 + 10.4r.$$

48. (a) Reading coordinates from the graph, we see that rainfall $r = 100$ mm corresponds to about $Q = 300$ kg/hectare, and $r = 600$ mm corresponds to about $Q = 3200$ kg/hectare. Using a difference quotient we have

$$\text{Slope} = \frac{\Delta Q}{\Delta r} = \frac{3200 - 300}{600 - 100} = 5.8 \text{ kg/hectare per mm.}$$

(b) Every additional 1 mm of annual rainfall corresponds to an additional 5.8 kg of grass per hectare.

(c) Using the slope, we see that the equation has the form

$$Q = b + 5.8r.$$

Substituting $r = 100$ and $Q = 300$ we can solve for b.

$$\begin{aligned} b + 5.8(100) &= 300 \\ b &= -280. \end{aligned}$$

The equation of the line is

$$Q = -280 + 5.8r.$$

49. The difference quotient $\Delta Q/\Delta r$ equals the slope of the line and represents the increase in the quantity of grass per millimeter of rainfall. We see from the graph that the slope of the line for 1939 is larger than the slope of the line for 1997. Thus, each additional 1 mm of rainfall in 1939 led to a larger increase in the quantity of grass than in 1997.

50. (a) Let x be the average minimum daily temperature (°C), and let y be the date the marmot is first sighted. Reading coordinates from the graph, we see that temperature $x = 12$°C corresponds to date $y = 137$ days after Jan 1, and $x = 22$°C corresponds to $y = 109$ days. Using a difference quotient, we have

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{109 - 137}{22 - 12} = -2.8 \, \frac{\text{days}}{^\circ\text{C}}.$$

(b) The slope is negative. An increase in the temperatures corresponds to an earlier sighting of a marmot. Marmots come out of hibernation earlier in years with warmer average daily minimum temperature.

(c) We have

$$\Delta y = \text{Slope} \times \Delta x = (-2.8)(6) = 16.8 \text{ days.}$$

If temperatures are 6°C higher, then marmots come out of hibernations about 17 days earlier.

(d) Using the slope, we see that the equation has the form

$$y = b - 2.8x.$$

Substituting $x = 12$ and $y = 137$ we solve for b.

$$\begin{aligned} b - 2.8(12) &= 137 \\ b &= 171. \end{aligned}$$

The equation of the line is

$$y = 171 - 2.8x.$$

51. (a) If the first date of bare ground is 140, then, according to the figure, the first bluebell flower is sighted about day 150, that is $150 - 140 = 10$ days later.

(b) Let x be the first date of bare ground, and let y be the date the first bluebell flower is sighted. Reading coordinates from the graph, we see that date $x = 130$ days after Jan 1 corresponds to date $y = 142$ days after Jan 1, and $x = 170$ days corresponds to $y = 173$ days. Using a difference quotient we have

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{173 - 142}{170 - 130} = 0.775 \text{ days per day.}$$

(c) The slope is positive, so an increase in the x-variable corresponds to an increase in the y-variable. These variables represents days in the year, and larger values indicate days that are later in the year. Thus if bare ground first occurs later in the year, then bluebells first flower later in the year. Positive slope means that bluebells flower later when the snow cover lasts longer.

(d) Using the slope, we see that the equation has the form

$$y = b + 0.775x.$$

Substituting $x = 130$ and $y = 142$ we can solve for b.

$$\begin{aligned} b + 0.775(130) &= 142 \\ b &= 41.25. \end{aligned}$$

The equation of the line is

$$y = 41.25 + 0.775x.$$

52. (a) We have $m = \dfrac{256}{18} = 14.222$ cm per hour. When the snow started, there were 100 cm on the ground, so

$$f(t) = 100 + 14.222t.$$

(b) The domain of f is $0 \le t \le 18$ hours. The range is $100 \le f(t) \le 356$ cm.

53. (a) We find the slope m and intercept b in the linear equation $C = b + mw$. To find the slope m, we use

$$m = \frac{\Delta C}{\Delta w} = \frac{12.32 - 8}{68 - 32} = 0.12 \text{ dollars per gallon.}$$

We substitute to find b:

$$\begin{aligned} C &= b + mw \\ 8 &= b + (0.12)(32) \\ b &= 4.16 \text{ dollars.} \end{aligned}$$

The linear formula is $C = 4.16 + 0.12w$.

(b) The slope is 0.12 dollars per gallon. Each additional gallon of waste collected costs 12 cents.

(c) The intercept is \$4.16. The flat monthly fee to subscribe to the waste collection service is \$4.16. This is the amount charged even if there is no waste.

54. We are looking for a linear function $y = f(x)$ that, given a time x in years, gives a value y in dollars for the value of the refrigerator. We know that when $x = 0$, that is, when the refrigerator is new, $y = 950$, and when $x = 7$, the refrigerator is worthless, so $y = 0$. Thus $(0, 950)$ and $(7, 0)$ are on the line that we are looking for. The slope is then given by

$$m = \frac{950}{-7}$$

It is negative, indicating that the value decreases as time passes. Having found the slope, we can take the point $(7, 0)$ and use the point-slope formula:

$$y - y_1 = m(x - x_1).$$

So,

$$\begin{aligned} y - 0 &= -\frac{950}{7}(x - 7) \\ y &= -\frac{950}{7}x + 950. \end{aligned}$$

55. (a) Charge per cubic foot $= \dfrac{\Delta \$}{\Delta \text{ cu. ft.}} = \dfrac{55-40}{1600-1000} = \$0.025/\text{cu. ft.}$
Alternatively, if we let c = cost, w = cubic feet of water, b = fixed charge, and m = cost/cubic feet, we obtain $c = b + mw$. Substituting the information given in the problem, we have

$$40 = b + 1000m$$
$$55 = b + 1600m.$$

Subtracting the first equation from the second yields $15 = 600m$, so $m = 0.025$.

(b) The equation is $c = b + 0.025w$, so $40 = b + 0.025(1000)$, which yields $b = 15$. Thus the equation is $c = 15 + 0.025w$.

(c) We need to solve the equation $100 = 15 + 0.025w$, which yields $w = 3400$. It costs \$100 to use 3400 cubic feet of water.

56. (a) We find the slope m and intercept b in the linear equation $S = b + mt$. To find the slope m, we use

$$m = \frac{\Delta S}{\Delta t} = \frac{66-113}{50-0} = -0.94.$$

When $t = 0$, we have $S = 113$, so the intercept b is 113. The linear formula is

$$S = 113 - 0.94t.$$

(b) We use the formula $S = 113 - 0.94t$. When $S = 20$, we have $20 = 113 - 0.94t$ and so $t = 98.9$. If this linear model were correct, the average male sperm count would drop below the fertility level during the year 2038.

57. (a) (i) $f(2001) = 272$
(ii) $f(2014) = 525$

(b) The average yearly increase is the rate of change.

$$\text{Yearly increase } = \frac{f(2014) - f(2001)}{2014 - 2001} = \frac{525 - 272}{13} = 19.46 \text{ billionaires per year.}$$

(c) Since we assume the rate of increase remains constant, we use a linear function with slope 19.46 billionaires per year. The equation is

$$f(t) = b + 19.46t$$

where $f(2001) = 272$, so

$$272 = b + 19.46(2001)$$
$$b = -38{,}667.5.$$

Thus, $f(t) = 19.46t - 38{,}667.5$.

58. (a) The variable costs for x acres are \$200x, or $0.2x$ thousand dollars. The total cost, C (again in thousands of dollars), of planting x acres is:

$$C = f(x) = 10 + 0.2x.$$

This is a linear function. See Figure 1.8. Since $C = f(x)$ increases with x, f is an increasing function of x. Look at the values of C shown in the table; you will see that each time x increases by 1, C increases by 0.2. Because C increases at a constant rate as x increases, the graph of C against x is a line.

(b) See Figure 1.8 and Table 1.1.

Table 1.1 *Cost of planting seed*

x	C
0	10
2	10.4
3	10.6
4	10.8
5	11
6	11.2

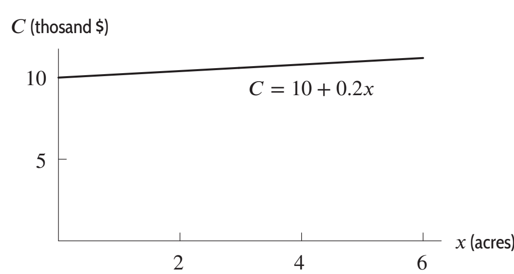


Figure 1.8

(c) The vertical intercept of 10 corresponds to the fixed costs. For $C = f(x) = 10 + 0.2x$, the intercept on the vertical axis is 10 because $C = f(0) = 10 + 0.2(0) = 10$. Since 10 is the value of C when $x = 0$, we recognize it as the initial outlay for equipment, or the fixed cost.

The slope 0.2 corresponds to the variable costs. The slope is telling us that for every additional acre planted, the costs go up by 0.2 thousand dollars. The rate at which the cost is increasing is 0.2 thousand dollars per acre. Thus the variable costs are represented by the slope of the line $f(x) = 10 + 0.2x$.

59. We will let

$$\begin{aligned} T &= \text{ amount of fuel for take-off,} \\ L &= \text{ amount of fuel for landing,} \\ P &= \text{ amount of fuel per mile in the air,} \\ m &= \text{ the length of the trip in miles.} \end{aligned}$$

Then Q, the total amount of fuel needed, is given by

$$Q(m) = T + L + Pm.$$

60. (a) The scale of the graph makes it impossible to read values of $f(423)$ and $f(422)$ accurately enough to evaluate their difference. But that difference equals the slope of the line, which we can estimate. Using the points $(400, 2000)$ and $(600, 6000)$ on the line, we have

$$\text{Slope} = \frac{6000 - 2000}{600 - 400} = 20.$$

Thus $f(423) - f(422) = 20$.

(b) We have

$$\begin{aligned} \Delta y &= \text{Slope} \times \Delta x \\ &= (20)(517 - 513) = 80. \end{aligned}$$

Thus $f(517) - f(513) = 80$.

61. (a) The scale of the graph makes it impossible to read values of $g(4210)$ and $g(4209)$ accurately enough to evaluate their difference. But that difference equals the slope of the line, which we can estimate. Using the points $(3000, 70)$ and $(5000, 50)$ on the line, we have

$$\text{Slope} = \frac{50 - 70}{5000 - 3000} = -0.01.$$

So

$$\begin{aligned} \Delta y &= \text{Slope} \times \Delta x \\ &= (-0.01)(4210 - 4209) = -0.01. \end{aligned}$$

Thus $g(4210) - g(4209) = -0.01$.

(b) We have

$$\begin{aligned} \Delta y &= \text{Slope} \times \Delta x \\ &= (-0.01)(3760 - 3740) = -0.2. \end{aligned}$$

Thus $g(3760) - g(3740) = -0.2$.

62. (a) The largest time interval was 2010–2012 since the percentage growth rate increased from -60.5 to 69.9 from 2010 to 2012. This means the US exports of biofuels grew relatively more from 2010 to 2012 than from 2012 to 2013. (Note that the percentage growth rate was a decreasing function of time over 2012–2014.)

(b) The largest time interval was 2012–2013 since the percentage growth rates were positive for each of these two consecutive years. This means that the amount of biofuels exported from the US steadily increased during the two years from 2012 to 2013, then decreased in 2014.

63. (a) The largest time interval was 2007–2009 since the percentage growth rate increased from -14.6 in 2007 to 6.3 in 2009. This means that from 2007 to 2009 the US consumption of hydroelectric power grew relatively more with each successive year.

(b) The largest time interval was 2012–2014 since the percentage growth rates were negative for each of these three consecutive years. This means that the amount of hydroelectric power consumed by the US industrial sector steadily decreased during the three year span from 2012 to 2014.

64. (a) The largest time interval was 2008–2010 since the percentage growth rate decreased from 3.6 in 2008 to −29.7 in 2010. This means that from 2008 to 2010 the US price per watt of a solar panel fell relatively more with each successive year.

(b) The largest time interval was 2009–2010 since the percentage growth rates were negative for each of these two consecutive years. This means that the US price per watt of a solar panel decreased during the two year span from 2009 to 2010, after an increase in the previous year.

65. (a) Since 2008 corresponds to $t = 0$, the average annual sea level in Aberdeen in 2008 was 7.094 meters.

(b) Looking at the table, we see that the average annual sea level was 7.019 twenty five years before 2008, or in the year 1983. Similar reasoning shows that the average sea level was 6.957 meters 125 years before 2008, or in 1883.

(c) Because 125 years before 2008 the year was 1883, we see that the sea level value corresponding to the year 1883 is 6.957 (this is the sea level value corresponding to $t = 125$). Similar reasoning yields the table:

Year	1883	1908	1933	1958	1983	2008
S	6.957	6.938	6.965	6.992	7.019	7.094

66. (a) This could be a linear function because w increases by 2.7 as h increases by 4.

(b) We find the slope m and the intercept b in the linear equation $w = b + mh$. We first find the slope m using the first two points in the table. Since we want w to be a function of h, we take

$$m = \frac{\Delta w}{\Delta h} = \frac{82.4 - 79.7}{172 - 168} = 0.68.$$

Substituting the first point and the slope $m = 0.68$ into the linear equation $w = b+mh$, we have $79.7 = b+(0.68)(168)$, so $b = -34.54$. The linear function is

$$w = 0.68h - 34.54.$$

The slope, $m = 0.68$, is in units of kg per cm.

(c) We find the slope and intercept in the linear function $h = b + mw$ using $m = \Delta h/\Delta w$ to obtain the linear function

$$h = 1.47w + 50.79.$$

Alternatively, we could solve the linear equation found in part (b) for h. The slope, $m = 1.47$, has units cm per kg.

67. (a) The first company's price for a day's rental with m miles on it is $C_1(m) = 40+0.15m$. Its competitor's price for a day's rental with m miles on it is $C_2(m) = 50 + 0.10m$.

(b) See Figure 1.9.

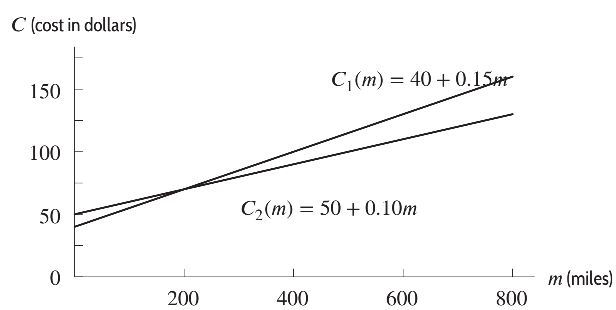


Figure 1.9

(c) To find which company is cheaper, we need to determine where the two lines intersect. We let $C_1 = C_2$, and thus

$$\begin{aligned} 40 + 0.15m &= 50 + 0.10m \\ 0.05m &= 10 \\ m &= 200. \end{aligned}$$

If you are going more than 200 miles a day, the competitor is cheaper. If you are going less than 200 miles a day, the first company is cheaper.

68. (a) Since the initial value is \$25,000 and the slope is −2000, the value of the vehicle at time t is

$$V(t) = 25000 - 2000t.$$

The cost of repairs has initial value 0 and slope 1500, so

$$C(t) = 1500t.$$

Figure 1.10 shows the graphs of these two functions.

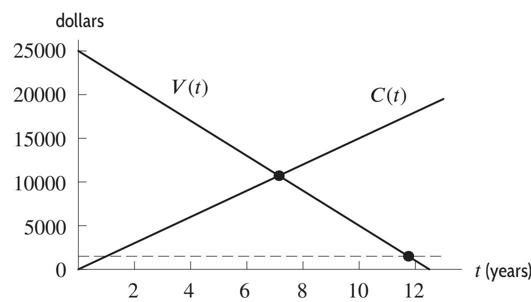


Figure 1.10

(b) The vehicle value equals the repair cost, when $V(t) = C(t)$.

$$\begin{aligned} 25000 - 2000t &= 1500t \\ 25000 &= 3500t \\ 7.143 &= t. \end{aligned}$$

Thus, replace during the eighth year. Since 0.143 years is 0.143(12) = 1.7 months, replacement should take place in the second month of the eighth year.

(c) Since 6% of the orginal value is 1500, the vehicle should be replaced when $V(t) = 1500$.

$$\begin{aligned} 25000 - 2000t &= 1500 \\ 25000 &= 1500 + 2000t \\ 23500 &= 2000t \\ 11.750 &= t. \end{aligned}$$

Thus, in the twelfth year. Since 0.75 years is 9 months, replacement should take place at the end of the ninth month of the twelfth year.

69. (a) If the bakery owner decreases the price, the customers want to buy more. Thus, the slope of $d(q)$ is negative. If the owner increases the price, she is make more cakes. Thus the slope of $s(q)$ is positive.

(b) To determine whether an ordered pair (q, p) is a solution to the inequality, we substitute the values of q and p into the inequality and see whether the resulting statement is true. Substituting the two ordered pairs gives

$(60, 18)$: $18 \le 20 - 60/20$, or $18 \le 17$. This is false, so $(60, 18)$ is not a solution.

$(120, 12)$: $12 \le 20 - 120/20$, or $12 \le 14$. This is true, so $(120, 12)$ is a solution.

The pair $(60, 18)$ is not a solution to the inequality $p \le 20 - q/20$. This means that the price \$18 is higher than the unit price at which customers would be willing to buy a total of 60 cakes. So customers are not willing to buy 60 cakes at \$18. The pair $(120, 12)$ is a solution, meaning that \$12 is not more than the price at which customers would be willing to buy 120 cakes. Thus customers are willing to buy a total of 120 (and more) cakes at \$12. Each solution (q, p) represents a quantity of cakes q that customers would be willing to buy at the unit price p.

(c) In order to be a solution to both of the given inequalities, a point (q, p) must lie on or below the line $p = 20 - q/20$ and on or above the line $p = 11 + q/40$. Thus, the solution set of the given system of inequalities is the region shaded in Figure 1.11.

A point (q, p) in this region represents a quantity q of cakes that customers would be willing to buy, and that the bakery-owner would be willing to make and sell, at the price p.

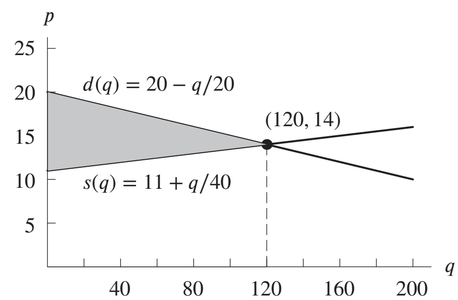


Figure 1.11: Possible cake sales at different prices and quantities

(d) To find the rightmost point of this region, we need to find the intersection point of the lines $p = 20 - q/20$ and $p = 11 + q/40$. At this point, p is equal to both $20 - q/20$ and $11 + q/40$, so these two expressions are equal to each other:

$$\begin{aligned} 20 - \frac{q}{20} &= 11 + \frac{q}{40} \\ 9 &= \frac{q}{20} + \frac{q}{40} \\ 9 &= \frac{3q}{40} \\ q &= 120. \end{aligned}$$

Therefore, $q = 120$ is the maximum number of cakes that can be sold at a price at which customers are willing to buy them all, and the owner of the bakery is willing to make them all. The price at this point is $p = 20 - 120/20 = 14$ dollars. (In economics, this price is called the *equilibrium price*, since at this point there is no incentive for the owner of the bakery to raise or lower the price of the cakes.)

70. (a) The line given by $(0, 2)$ and $(1, 1)$ has slope $m = \frac{2-1}{-1} = -1$ and y-intercept 2, so its equation is

$$y = -x + 2.$$

The points of intersection of this line with the parabola $y = x^2$ are given by

$$\begin{aligned} x^2 &= -x + 2 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0. \end{aligned}$$

The solution $x = 1$ corresponds to the point we are already given, so the other solution, $x = -2$, gives the x-coordinate of C. When we substitute back into either equation to get y, we get the coordinates for C, $(-2, 4)$.

(b) The line given by $(0, b)$ and $(1, 1)$ has slope $m = \frac{b-1}{-1} = 1 - b$, and y-intercept at $(0, b)$, so we can write the equation for the line as we did in part (a):

$$y = (1 - b)x + b.$$

We then solve for the points of intersection with $y = x^2$ the same way:

$$\begin{aligned} x^2 &= (1 - b)x + b \\ x^2 - (1 - b)x - b &= 0 \\ x^2 + (b - 1)x - b &= 0 \\ (x + b)(x - 1) &= 0 \end{aligned}$$

Again, we have the solution at the given point $(1, 1)$, and a new solution at $x = -b$, corresponding to the other point of intersection C. Substituting back into either equation, we can find the y-coordinate for C is b^2, and thus C is given by $(-b, b^2)$. This result agrees with the particular case of part (a) where $b = 2$.

71. Looking at the given data, it seems that Galileo's hypothesis was incorrect. The first table suggests that velocity is not a linear function of distance, since the increases in velocity for each foot of distance are themselves getting smaller. Moreover, the second table suggests that velocity is instead proportional to *time*, since for each second of time, the velocity increases by 32 ft/sec.

Strengthen Your Understanding

72. The slope of a linear function of x is given by the function's rise (Δy) over its run (Δx) over any interval. So, for $y = b+mx$, we have:

$$\text{Slope} = m = \frac{\Delta y}{\Delta x}.$$

73. The line $x = 3$ is vertical, so it is not a function of x: it fails the vertical line test. The line $y = 3$ is horizontal, hence a linear function of x with slope 0.

74. The line $y - 3 = 0$ or, equivalently, $y = 3$ is horizontal, hence has slope 0 in the xy-plane.

75. The line $y = 0.5 - 3x$ has a negative slope and is therefore a decreasing function.

76. If y is directly proportional to x we have $y = kx$. Adding the constant 1 to give $y = 2x + 1$ means that y is not proportional to x.

77. One possible answer is $f(x) = 2x + 3$.

78. One possible answer is $q = \dfrac{8}{p^{1/3}}$.

79. False. A line can be put through any two points in the plane. However, if the line is vertical, it is not the graph of a function.

80. True. Suppose we start at $x = x_1$ and increase x by 1 unit to $x_1 + 1$. If $y = b + mx$, the corresponding values of y are $b + mx_1$ and $b + m(x_1 + 1)$. Thus y increases by

$$\Delta y = b + m(x_1 + 1) - (b + mx_1) = m.$$

81. True. Solving for y on the second equation, we see that the second linear function has the same equation as the first:

$$\begin{aligned} x &= -y + 1 \\ x - 1 &= -y \\ -x + 1 &= y. \end{aligned}$$

82. False. Solving for y on the second equation, we get $y = (-1/2)x + 1$ which is a linear function with a different slope and intercept from the first equation.

83. False. For example, let $y = x + 1$. Then the points $(1, 2)$ and $(2, 3)$ are on the line. However the ratios

$$\frac{2}{1} = 2 \qquad \text{and} \qquad \frac{3}{2} = 1.5$$

are different. The ratio y/x is constant for linear functions of the form $y = mx$, but not in general. (Other examples are possible.)

84. False. For example, if $y = 4x + 1$ (so $m = 4$) and $x = 1$, then $y = 5$. Increasing x by 2 units gives 3, so $y = 4(3) + 1 = 13$. Thus, y has increased by 8 units, not $4 + 2 = 6$. (Other examples are possible.)

85. (b) and (c). For $g(x) = \sqrt{x}$, the domain and range are all nonnegative numbers, and for $h(x) = x^3$, the domain and range are all real numbers.

Solutions for Section 1.2

Exercises

1. The graph shows a concave up function.

2. The graph shows a concave down function.

3. This graph is neither concave up or down.

4. The graph is concave up.

5. Initial quantity = 5; growth rate = 0.07 = 7%.

6. Initial quantity = 7.7; growth rate = −0.08 = −8% (decay).

7. Initial quantity = 3.2; growth rate = 0.03 = 3% (continuous).

8. Initial quantity = 15; growth rate = −0.06 = −6% (continuous decay).

9. Since $e^{0.25t} = \left(e^{0.25}\right)^t \approx (1.2840)^t$, we have $P = 15(1.2840)^t$. This is exponential growth since 0.25 is positive. We can also see that this is growth because $1.2840 > 1$.

10. Since $e^{-0.5t} = (e^{-0.5})^t \approx (0.6065)^t$, we have $P = 2(0.6065)^t$. This is exponential decay since −0.5 is negative. We can also see that this is decay because $0.6065 < 1$.

11. $P = P_0(e^{0.2})^t = P_0(1.2214)^t$. Exponential growth because $0.2 > 0$ or $1.2214 > 1$.

12. $P = 7(e^{-\pi})^t = 7(0.0432)^t$. Exponential decay because $-\pi < 0$ or $0.0432 < 1$.

13. (a) Let $Q = Q_0a^t$. Then $Q_0a^5 = 75.94$ and $Q_0a^7 = 170.86$. So

$$\frac{Q_0a^7}{Q_0a^5} = \frac{170.86}{75.94} = 2.25 = a^2.$$

So $a = 1.5$.

(b) Since $a = 1.5$, the growth rate is $r = 0.5 = 50\%$.

14. (a) Let $Q = Q_0a^t$. Then $Q_0a^{0.02} = 25.02$ and $Q_0a^{0.05} = 25.06$. So

$$\frac{Q_0a^{0.05}}{Q_0a^{0.02}} = \frac{25.06}{25.02} = 1.001 = a^{0.03}.$$

So

$$a = (1.001)^{\frac{100}{3}} = 1.05.$$

(b) Since $a = 1.05$, the growth rate is $r = 0.05 = 5\%$.

15. (a) The function is linear with initial population of 1000 and slope of 50, so $P = 1000 + 50t$.

(b) This function is exponential with initial population of 1000 and growth rate of 5%, so $P = 1000(1.05)^t$.

16. (a) This is a linear function with slope −2 grams per day and intercept 30 grams. The function is $Q = 30 - 2t$, and the graph is shown in Figure 1.12.

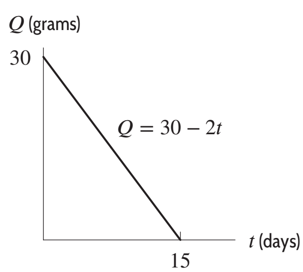


Figure 1.12

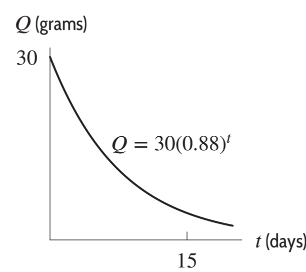


Figure 1.13

(b) Since the quantity is decreasing by a constant percent change, this is an exponential function with base $1-0.12 = 0.88$. The function is $Q = 30(0.88)^t$, and the graph is shown in Figure 1.13.

17. The function is increasing and concave up between D and E, and between H and I. It is increasing and concave down between A and B, and between E and F. It is decreasing and concave up between C and D, and between G and H. Finally, it is decreasing and concave down between B and C, and between F and G

18. (a) It was decreasing from March 2 to March 5 and increasing from March 5 to March 9.

(b) From March 5 to 8, the average temperature increased, but the rate of increase went down, from 12° between March 5 and 6 to 4° between March 6 and 7 to 2° between March 7 and 8.

From March 7 to 9, the average temperature increased, and the rate of increase went up, from 2° between March 7 and 8 to 9° between March 8 and 9.

Problems

19. (a) A linear function must change by exactly the same amount whenever x changes by some fixed quantity. While $h(x)$ decreases by 3 whenever x increases by 1, $f(x)$ and $g(x)$ fail this test, since both change by different amounts between

$x = -2$ and $x = -1$ and between $x = -1$ and $x = 0$. So the only possible linear function is $h(x)$, so it will be given by a formula of the type: $h(x) = mx + b$. As noted, $m = -3$. Since the y-intercept of h is 31, the formula for $h(x)$ is $h(x) = 31 - 3x$.

(b) An exponential function must grow by exactly the same factor whenever x changes by some fixed quantity. Here, $g(x)$ increases by a factor of 1.5 whenever x increases by 1. Since the y-intercept of $g(x)$ is 36, $g(x)$ has the formula $g(x) = 36(1.5)^x$. The other two functions are not exponential; $h(x)$ is not because it is a linear function, and $f(x)$ is not because it both increases and decreases.

20. Table A and Table B could represent linear functions of x. Table A could represent the constant linear function $y = 2.2$ because all y values are the same. Table B could represent a linear function of x with slope equal to $11/4$. This is because x values that differ by 4 have corresponding y values that differ by 11, and x values that differ by 8 have corresponding y values that differ by 22. In Table C, y decreases and then increases as x increases, so the table cannot represent a linear function. Table D does not show a constant rate of change, so it cannot represent a linear function.

21. Table D is the only table that could represent an exponential function of x. This is because, in Table D, the ratio of y values is the same for all equally spaced x values. Thus, the y values in the table have a constant percent rate of decrease:

$$\frac{9}{18} = \frac{4.5}{9} = \frac{2.25}{4.5} = 0.5.$$

Table A represents a constant function of x, so it cannot represent an exponential function. In Table B, the ratio between y values corresponding to equally spaced x values is not the same. In Table C, y decreases and then increases as x increases. So neither Table B nor Table C can represent exponential functions.

22. Since f is linear, its slope is a constant

$$m = \frac{20 - 10}{2 - 0} = 5.$$

Thus f increases 5 units for unit increase in x, so

$$f(1) = 15, \quad f(3) = 25, \quad f(4) = 30.$$

Since g is exponential, its growth factor is constant. Writing $g(x) = ab^x$, we have $g(0) = a = 10$, so

$$g(x) = 10 \cdot b^x.$$

Since $g(2) = 10 \cdot b^2 = 20$, we have $b^2 = 2$ and since $b > 0$, we have

$$b = \sqrt{2}.$$

Thus g increases by a factor of $\sqrt{2}$ for unit increase in x, so

$$g(1) = 10\sqrt{2}, \quad g(3) = 10(\sqrt{2})^3 = 20\sqrt{2}, \quad g(4) = 10(\sqrt{2})^4 = 40.$$

Notice that the value of $g(x)$ doubles between $x = 0$ and $x = 2$ (from $g(0) = 10$ to $g(2) = 20$), so the doubling time of $g(x)$ is 2. Thus, $g(x)$ doubles again between $x = 2$ and $x = 4$, confirming that $g(4) = 40$.

23. We see that $\frac{1.09}{1.06} \approx 1.03$, and therefore $h(s) = c(1.03)^s$; c must be 1. Similarly $\frac{2.42}{2.20} = 1.1$, and so $f(s) = a(1.1)^s$; $a = 2$. Lastly, $\frac{3.65}{3.47} \approx 1.05$, so $g(s) = b(1.05)^s$; $b \approx 3$.

24. (a) This is the graph of a linear function, which increases at a constant rate, and thus corresponds to $k(t)$, which increases by 0.3 over each interval of 1.

(b) This graph is concave down, so it corresponds to a function whose increases are getting smaller, as is the case with $h(t)$, whose increases are 10, 9, 8, 7, and 6.

(c) This graph is concave up, so it corresponds to a function whose increases are getting bigger, as is the case with $g(t)$, whose increases are 1, 2, 3, 4, and 5.

25. (a) This is a linear function, corresponding to $g(x)$, whose rate of decrease is constant, 0.6.

(b) This graph is concave down, so it corresponds to a function whose rate of decrease is increasing, like $h(x)$. (The rates are $-0.2, -0.3, -0.4, -0.5, -0.6$.)

(c) This graph is concave up, so it corresponds to a function whose rate of decrease is decreasing, like $f(x)$. (The rates are $-10, -9, -8, -7, -6$.)

26. Graph (I) and (II) are of increasing functions, therefore the growth factor for both functions is greater than 1. Since graph (I) increases faster than graph (II), graph (I) corresponds to the larger growth factor. Therefore graph (I) matches $Q = 50(1.4)^t$ and graph (II) is $Q = 50(1.2)^t$.

Similarly, since graphs (III) and (IV) are both of decreasing functions, they have growth factors between 0 and 1. Since graph (IV) decreases faster than graph (III), graph (IV) has the smaller decay factor. Thus graph (III) is $Q = 50(0.8)^t$ and graph (IV) is $Q = 50(0.6)^t$.

27. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(0, 3)$ and $(2, 12)$ are on the graph, so

$$3 = y_0 a^0 = y_0$$

and

$$12 = y_0 \cdot a^2 = 3 \cdot a^2, \quad \text{giving} \quad a = \pm 2.$$

Since $a > 0$, our equation is $y = 3(2^x)$.

28. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(0, 18)$ and $(2, 8)$ are on the graph, so

$$18 = y_0 a^0 = y_0$$

and

$$8 = y_0 \cdot a^2 = 18 \cdot a^2, \quad \text{giving} \quad a = \pm\sqrt{\frac{4}{9}}.$$

Since $a > 0$, our equation is

$$y = 18\left(\frac{2}{3}\right)^x.$$

29. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(1, 6)$ and $(2, 18)$ are on the graph, so

$$6 = y_0 a^1 \quad \text{and} \quad 18 = y_0 a^2$$

Therefore $a = \frac{y_0 a^2}{y_0 a} = \dfrac{18}{6} = 3$, and so $6 = y_0 a = y_0 \cdot 3$; thus, $y_0 = 2$. Hence $y = 2(3^x)$.

30. The graphs looks like that of an exponential function that has been shifted down by 2 units since it appears that it has horizontal asymptote $y = -2$. Therefore we are looking for an equation of the form $y = ab^x - 2$. Since at $x = 0$ we have $y = 8 = a - 2$, giving $a = 10$. The point $(1, 4)$ is on the graph, so $4 = 10b^1 - 2$, giving $b = \frac{6}{10} = \frac{3}{5}$.
Therefore,

$$y = 10\left(\frac{3}{5}\right)^x - 2.$$

31. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(-1, 8)$ and $(1, 2)$ are on the graph, so

$$8 = y_0 a^{-1} \quad \text{and} \quad 2 = y_0 a^1$$

Therefore $\dfrac{8}{2} = \dfrac{y_0 a^{-1}}{y_0 a} = \dfrac{1}{a^2}$, giving $a = \frac{1}{2}$, and so $2 = y_0 a^1 = y_0 \cdot \frac{1}{2}$, so $y_0 = 4$.
Hence $y = 4\left(\frac{1}{2}\right)^x$.

32. The difference, D, between the horizontal asymptote and the graph appears to decrease exponentially, so we look for an equation of the form

$$D = D_0 a^x$$

where $D_0 = 4 =$ difference when $x = 0$. Since $D = 4 - y$, we have

$$4 - y = 4a^x \quad \text{or} \quad y = 4 - 4a^x = 4(1 - a^x)$$

The point $(1, 2)$ is on the graph, so $2 = 4(1 - a^1)$, giving $a = \frac{1}{2}$.
Therefore $y = 4(1 - (\frac{1}{2})^x)$.

33. (a) See Figure 1.14.
(b) No. The points in the plot do not lie even approximately on a straight line, so a linear model is a poor choice.
(c) No. The plot shows that H is a decreasing function of Y that might be leveling off to an asymptote at $H = 0$. These are features of an exponential decay function, but their presence does not show that H is an exponential function of Y. We can do a more precise check by dividing each value of H by the previous year's H.

$$\frac{\text{Houses in 2011}}{\text{Houses in 2010}} = \frac{13 \text{ million}}{18.3 \text{ million}} = 0.710$$
$$\frac{\text{Houses in 2012}}{\text{Houses in 2011}} = \frac{7.8 \text{ million}}{13 \text{ million}} = 0.600$$
$$\frac{\text{Houses in 2013}}{\text{Houses in 2012}} = \frac{3.9 \text{ million}}{7.8 \text{ million}} = 0.500$$

$$\frac{\text{Houses in 2014}}{\text{Houses in 2013}} = \frac{1\text{ million}}{3.9\text{ million}} = 0.256$$
$$\frac{\text{Houses in 2015}}{\text{Houses in 2014}} = \frac{0.5\text{ million}}{1\text{ million}} = 0.500.$$

If H were an exponential function of Y, the ratios would be approximately constant. Since they are not approximately constant, we see that an exponential model is a poor choice.

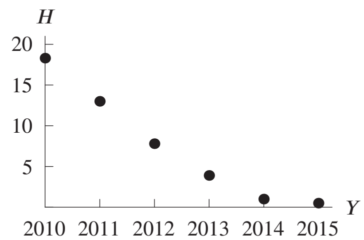


Figure 1.14

34. (a) See Figure 1.15.

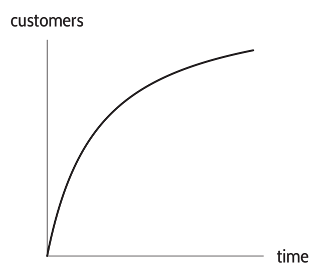


Figure 1.15

(b) "The rate at which new people try it" is the rate of change of the total number of people who have tried the product. Thus, the statement of the problem is telling you that the graph is concave down—the slope is positive but decreasing, as the graph shows.

35. (a) Advertising is generally cheaper in bulk; spending more money will give better and better marginal results initially, (Spending \$5,000 could give you a big newspaper ad reaching 200,000 people; spending \$100,000 could give you a series of TV spots reaching 50,000,000 people.) See Figure 1.16.

(b) The temperature of a hot object decreases at a rate proportional to the difference between its temperature and the temperature of the air around it. Thus, the temperature of a very hot object decreases more quickly than a cooler object. The graph is decreasing and concave up. See Figure 1.17 (We are assuming that the coffee is all at the same temperature.)

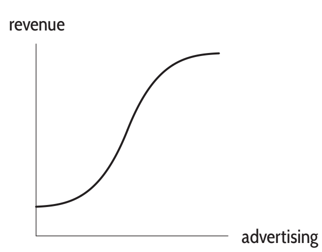


Figure 1.16

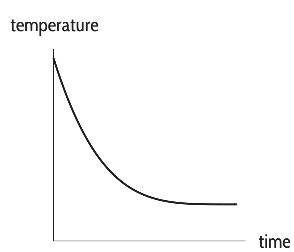
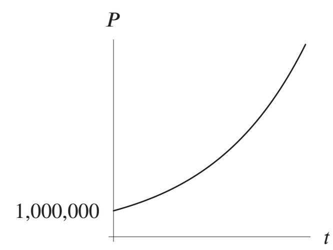


Figure 1.17

36. (a) We have $P_0 = 1$ million, and $k = 0.02$, so $P = (1{,}000{,}000)(e^{0.02t})$.
(b)



37. If production continues to decay exponentially at a continuous rate of 9.1% per year, the production $f(t)$ at time t years after 2008 is

$$f(t) = 84e^{-0.091t}.$$

In 2025, production is predicted to be

$$f(17) = 84e^{-0.091(17)} = 17.882 \text{ million barrels per day.}$$

38. (a) Let P represent the population of the world, and let t represent the number of years since 2014. Then we have $P = 7.17(1.011)^t$.
(b) According to this formula, the population of the world in the year 2020 (at $t = 6$) will be $P = 7.17(1.011)^6 = 7.66$ billion people.
(c) The graph is shown in Figure 1.18. The population of the world has doubled when $P = 14.34$; we see on the graph that this occurs at approximately $t = 63.4$. Under these assumptions, the doubling time of the world's population is about 63.4 years.

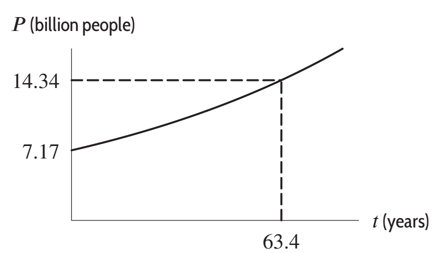


Figure 1.18

39. Direct calculation reveals that each 1000 foot increase in altitude results in a longer takeoff roll by a factor of about 1.096. Since the value of d when $h = 0$ (sea level) is $d = 670$, we are led to the formula

$$d = 670(1.096)^{h/1000},$$

where d is the takeoff roll, in feet, and h is the airport's elevation, in feet.

Alternatively, we can write

$$d = d_0 a^h,$$

where d_0 is the sea level value of d, $d_0 = 670$. In addition, when $h = 1000$, $d = 734$, so

$$734 = 670a^{1000}.$$

Solving for a gives

$$a = \left(\frac{734}{670}\right)^{1/1000} = 1.00009124,$$

so

$$d = 670(1.00009124)^h.$$

40. Since we are told that the rate of decay is *continuous*, we use the function $Q(t) = Q_0e^{rt}$ to model the decay, where $Q(t)$ is the amount of strontium-90 which remains at time t, and Q_0 is the original amount. Then

$$Q(t) = Q_0e^{-0.0247t}.$$

So after 100 years,

$$Q(100) = Q_0e^{-0.0247\cdot 100}$$

and

$$\frac{Q(100)}{Q_0} = e^{-2.47} \approx 0.0846$$

so about 8.46% of the strontium-90 remains.

41. (a) Since y is the number of US colonies, we need a growth factor smaller than 1 in order to describe a colony loss. A loss of 18.9% per year means a yearly growth factor equal to

$$1 - 0.189 = 0.811.$$

An annual percentage loss smaller than 18.9% would still be sustainable, and would result in a growth factor larger than 0.811 but still smaller than 1. Hence, II is the only function that could describe an economically sustainable colony loss trend.

(b) Function I says the number of US bee colonies is growing by 18.9% a year. Function II says bee colony loss is 10.9% per year. Function III indicates a bee colony loss of 20.2% a year.

42. The doubling time t depends only on the growth rate; it is the solution to

$$2 = (1.02)^t,$$

since 1.02^t represents the factor by which the population has grown after time t. Trial and error shows that $(1.02)^{35} \approx 1.9999$ and $(1.02)^{36} \approx 2.0399$, so that the doubling time is about 35 years.

43. (a) Since sales of electronic devices doubled in 12 years, sales in 1997 were $438/2 = 219$ million devices. Because part (b) asks for the annual percentage growth rate (not a continuous growth rate), we use an exponential function of the form $S(t) = S_0b^t$, where b is the annual growth factor. We have

$$438 = 219b^{12},$$

so $2 = b^{12}$ and $b = 2^{1/12} = 1.05946$. Thus,

$$S(t) = 219(1.05946)^t.$$

(b) The annual growth rate was 5.946%.

44. (a) Because the population is growing exponentially, the time it takes to double is the same, regardless of the population levels we are considering. For example, the population is 20,000 at time 3.7, and 40,000 at time 6.0. This represents a doubling of the population in a span of $6.0 - 3.7 = 2.3$ years.

How long does it take the population to double a second time, from 40,000 to 80,000? Looking at the graph once again, we see that the population reaches 80,000 at time $t = 8.3$. This second doubling has taken $8.3 - 6.0 = 2.3$ years, the same amount of time as the first doubling.

Further comparison of any two populations on this graph that differ by a factor of two will show that the time that separates them is 2.3 years. Similarly, during any 2.3 year period, the population will double. Thus, the doubling time is 2.3 years.

(b) Suppose $P = P_0a^t$ doubles from time t to time $t+d$. We now have $P_0a^{t+d} = 2P_0a^t$, so $P_0a^ta^d = 2P_0a^t$. Thus, canceling P_0 and a^t, d must be the number such that $a^d = 2$, no matter what t is.

45. (a) After 50 years, the amount of money is

$$P = 2P_0.$$

After 100 years, the amount of money is

$$P = 2(2P_0) = 4P_0.$$

After 150 years, the amount of money is

$$P = 2(4P_0) = 8P_0.$$

(b) The amount of money in the account doubles every 50 years. Thus in t years, the balance doubles $t/50$ times, so

$$P = P_02^{t/50}.$$

46. (a) Since $162.5 = 325/2$, there are 162.5 mg remaining after H hours.
Since $81.25 = 162.5/2$, there are 81.25 mg remaining H hours after there were 162.5 mg, so $2H$ hours after there were 325 mg.
Since $40.625 = 81.25/2$, there are 41.625 mg remaining H hours after there were 81.25 mg, so $3H$ hours after there were 325 mg.

(b) Each additional H hours, the quantity is halved. Thus in t hours, the quantity was halved t/H times, so

$$A = 325\left(\frac{1}{2}\right)^{t/H}.$$

47. (a) The quantity of radium decays exponentially, so we know $Q = Q_0 a^t$. When $t = 1620$, we have $Q = Q_0/2$ so

$$\frac{Q_0}{2} = Q_0 a^{1620}.$$

Thus, canceling Q_0, we have

$$\begin{aligned} a^{1620} &= \frac{1}{2} \\ a &= \left(\frac{1}{2}\right)^{1/1620} = 0.999572 \end{aligned}$$

Thus the formula is $Q = Q_0\,(0.999572)^t$.

(b) After 500 years,

$$\text{Fraction remaining} = \frac{1}{Q_0}\cdot Q_0\,(0.999572)^{500} = 0.80731.$$

so 80.731% is left.

48. Let Q_0 be the initial quantity absorbed in 1960. Then the quantity, Q, of strontium-90 left after t years is

$$Q = Q_0\left(\frac{1}{2}\right)^{t/29}.$$

Since $2010 - 1960 = 50$ years, in 2010

$$\text{Fraction remaining} = \frac{1}{Q_0}\cdot Q_0\left(\frac{1}{2}\right)^{50/29} = \left(\frac{1}{2}\right)^{50/29} = 0.30268 = 30.268\%.$$

49. (a) The 1.3 represents the number of food bank users, in thousands, when $t = 0$, that is in 2006.
(b) The continuous growth rate is 0.81, or 81%, per year.
(c) Since $e^{0.81t} = (e^{0.81})^t = 2.248^t$, we see that the annual growth factor is 2.248. Thus, the annual growth rate is $2.248-1 = 1.248 = 124.8\%$ per year.
(d) Since the annual growth rate is over 100% and the annual growth factor is 2.248, the number of users more than doubles each year. The doubling time is less than one year.

50. (a) Since the annual growth factor from 2009 to 2010 was $1 + (-0.193) = 0.807$ and $322(0.807) = 259.854$, the US consumed about 260 million gallons of biodiesel in 2010. Since the annual growth factor from 2010 to 2011 was $1 + 2.408 = 3.408$ and $260(3.408) = 886.08$, the US consumed approximately 886 million gallons of biodiesel in 2011.
(b) Completing the table of annual consumption of biodiesel and plotting the data gives Figure 1.19.

Year	2009	2010	2011	2012	2013	2014
Consumption of biodiesel (mn gal)	322	260	886	895	1404	1419

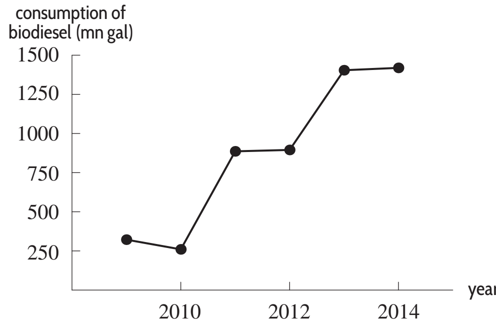


Figure 1.19

51. (a) False, because the annual percent growth is not constant over this interval.
(b) The US consumption of biodiesel more than tripled in 2011, since the annual percent growth in 2011 was over 200%.

52. (a) Since the annual growth factor from 2009 to 2010 was $1 - 0.049 = 0.951$ and $2.65(1 - 0.049) = 2.520$, the US consumed approximately 2.5 quadrillion BTUs of hydroelectric power in 2010. Since the annual growth factor from 2010 to 2011 was $1 + 0.224 = 1.224$ and $2.520(1 + 0.224) = 3.084$, the US consumed about 3.1 quadrillion BTUs of hydroelectric power in 2011.
(b) Completing the table of annual consumption of hydroelectric power and plotting the data gives Figure 1.20.

Year	2009	2010	2011	2012	2013	2014
Consumption of hydro. power (quadrillion BTU)	2.650	2.520	3.084	2.606	2.528	2.442

(c) The largest decrease in the US consumption of hydroelectric power occurred in 2011. In this year, the US consumption of hydroelectric power dropped by about 478 trillion BTUs to 2.606 quadrillion BTUs, down from 3.084 quadrillion BTUs in 2011.

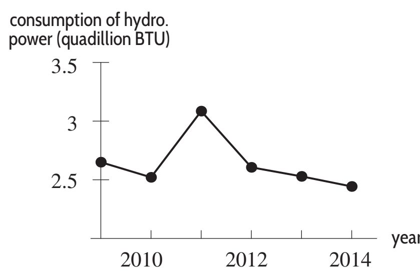


Figure 1.20

53. (a) From the figure we can read-off the approximate percent growth for each year over the previous year:

Year	2009	2010	2011	2012	2013	2014
% growth over previous yr	32	28	26	15	20	8

Since the annual growth factor from 2009 to 2010 was $1 + 0.28 = 1.28$ and $721(1 + 0.28) = 922.88$, the US consumed approximately 923 trillion BTUs of wind power energy in 2010. Since the annual growth factor from 2010 to 2011 was $1 + 0.26 = 1.26$ and $922.88(1 + 0.26) = 1162.829$, the US consumed about 1163 trillion BTUs of wind power energy in 2010.
(b) Completing the table of annual consumption of wind power and plotting the data gives Figure 1.21.

Year	2009	2010	2011	2012	2013	2014
Consumption of wind power (trillion BTU)	721	923	1163	1337	1605	1733

(c) The largest increase in the US consumption of wind power energy occurred in 2013. In this year the US consumption of wind power energy rose by about 268 trillion BTUs to 1605 trillion BTUs, up from 1337 trillion BTUs in 2012.

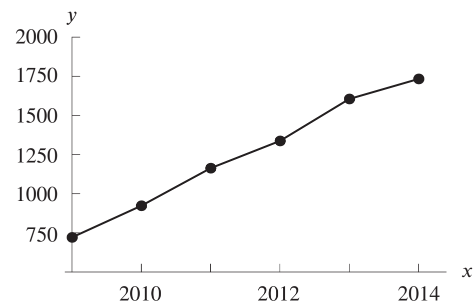


Figure 1.21

54. (a) The US consumption of wind power energy increased by at least 25% in 2009, 2010 and in 2011, relative to the previous year. Consumption did not decrease during the time period shown because all the annual percent growth values are positive, indicating a steady increase in the US consumption of wind power energy between 2009 and 2014.

(b) Yes. From 2008 to 2009 consumption increased by about 32%, which means $x(1+0.32)$ units of wind power energy were consumed in 2009 if x had been consumed in 2008. Similarly, $(x(1+0.32))(1+0.28)$ units of wind power energy were consumed in 2010 if x had been consumed in 2008 (because consumption increased by about 28% from 2009 to 2010). Repeating this process,

$$(x(1+0.32))(1+0.28)(1+0.26)$$

units of wind power energy were consumed in 2011 if x had been consumed in 2008 (because consumption increased by about 26% from 2010 to 2011). Since

$$(x(1+0.32))(1+0.28)(1+0.26) = x(2.129) = x(1+1.129),$$

the percent growth in wind power consumption was about 112.9%, or over 100%, in 2011 relative to consumption in 2008. Note that adding these three percents would be 86%, however there is a multiplier effect which makes the change well over 100%.

55. (a) Since the y-intercept of each graph is above the x-axis, we know that a, c, and p must all be positive.

(b) Since the graph of $y = ab^x$ is increasing and goes through $(1, 1)$, the y-intercept, a, must be less than 1. Since the y-intercept is positive, we know that $0 < a < 1$. Since $y = cd^x$ and $y = pq^x$ are both decreasing, both d and q must be between 0 and 1.

(c) The graphs of $y = cd^x$ and $y = pq^x$ have the same y-intercepts, therefore we have $c = p$.

(d) Since the point $(1, 1)$ lies on the graph of $y = ab^x$, we know that $1 = ab^1$. Therefore, a and b are reciprocals of each other. Similarly, p and q are reciprocals of each other.

Strengthen Your Understanding

56. A quantity that doubles daily has a daily growth factor of 2, and a daily growth rate of 100%.

57. The function $y = e^{-0.25x}$ is decreasing but its graph is concave up.

58. The graph of $y = 2x$ is a straight line and is neither concave up or concave down.

59. The last two points are not on the graph of e^x. Points of the graph of this function have the form (x, e^x). The values $x = 2$ and $x = 3$ correspond to the points $(2, e^2)$ and $(3, e^3)$.

60. The functions $y = e^{-t}$ and $y = (0.9)^t$, for example, are both decreasing but have a vertical intercept of 1. To get a vertical intercept of π we must get this value when evaluating at $t = 0$. Hence, two possible functions are $y = \pi e^{-t}$ and $y = \pi(0.9)^t$.

61. One possible answer is $q = 2.2(0.97)^t$.

62. One possible answer is $f(x) = 2(1.1)^x$.

63. One possibility is $y = e^{-x} - 5$.

64. Any exponential function of the form $y = a^x$, where $1 < a < e$ works. A possible example is $y = 2^x$.

65. True. Using algebra and rules of exponents, we can rewrite $f(x)$ so that it has the form P_0e^{kx}:

$$\begin{aligned} f(x) &= e^{2x}/(2e^{5x}) \\ &= \frac{1}{2}e^{2x-5x} \\ &= \frac{1}{2}e^{-3x}. \end{aligned}$$

66. False. The y-intercept is $y = 2 + 3e^{-0} = 5$.

67. True, since, as $t \to \infty$, we know $e^{-4t} \to 0$, so $y = 5 - 3e^{-4t} \to 5$.

68. False. Suppose $y = 5^x$. Then increasing x by 1 increases y by a factor of 5. However increasing x by 2 increases y by a factor of 25, not 10, since

$$y = 5^{x+2} = 5^x \cdot 5^2 = 25 \cdot 5^x.$$

(Other examples are possible.)

69. True. Suppose $y = Ab^x$ and we start at the point (x_1, y_1), so $y_1 = Ab^{x_1}$. Then increasing x_1 by 1 gives $x_1 + 1$, so the new y-value, y_2, is given by

$$y_2 = Ab^{x_1+1} = Ab^{x_1}b = (Ab^{x_1})b,$$

so

$$y_2 = by_1.$$

Thus, y has increased by a factor of b, so $b = 3$, and the function is $y = A3^x$.

However, if x_1 is increased by 2, giving $x_1 + 2$, then the new y-value, y_3, is given by

$$y_3 = A3^{x_1+2} = A3^{x_1}3^2 = 9A3^{x_1} = 9y_1.$$

Thus, y has increased by a factor of 9.

70. True. For example, $f(x) = (0.5)^x$ is an exponential function which decreases. (Other examples are possible.)

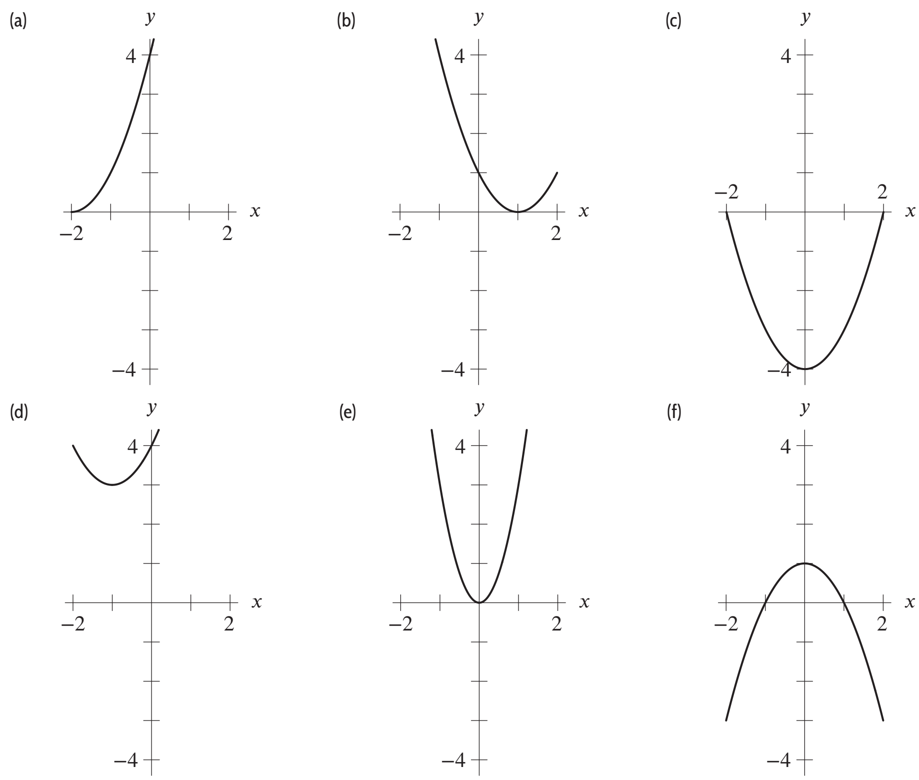
71. True. If $b > 1$, then $ab^x \to 0$ as $x \to -\infty$. If $0 < b < 1$, then $ab^x \to 0$ as $x \to \infty$. In either case, the function $y = a + ab^x$ has $y = a$ as the horizontal asymptote.

72. True, since $e^{-kt} \to 0$ as $t \to \infty$, so $y \to 20$ as $t \to \infty$.

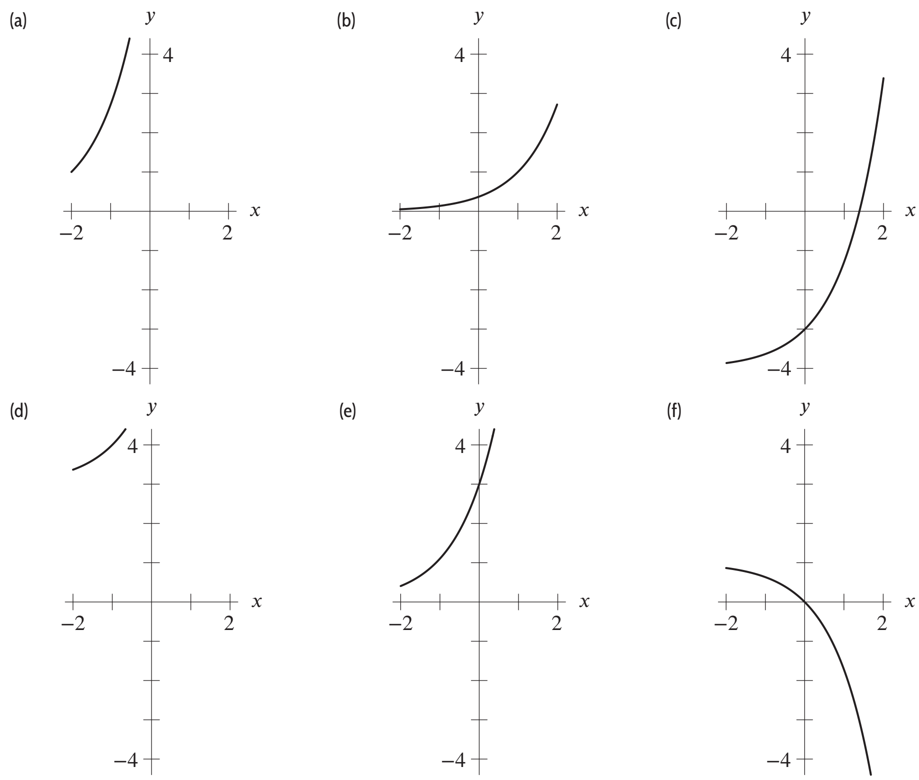
Solutions for Section 1.3

Exercises

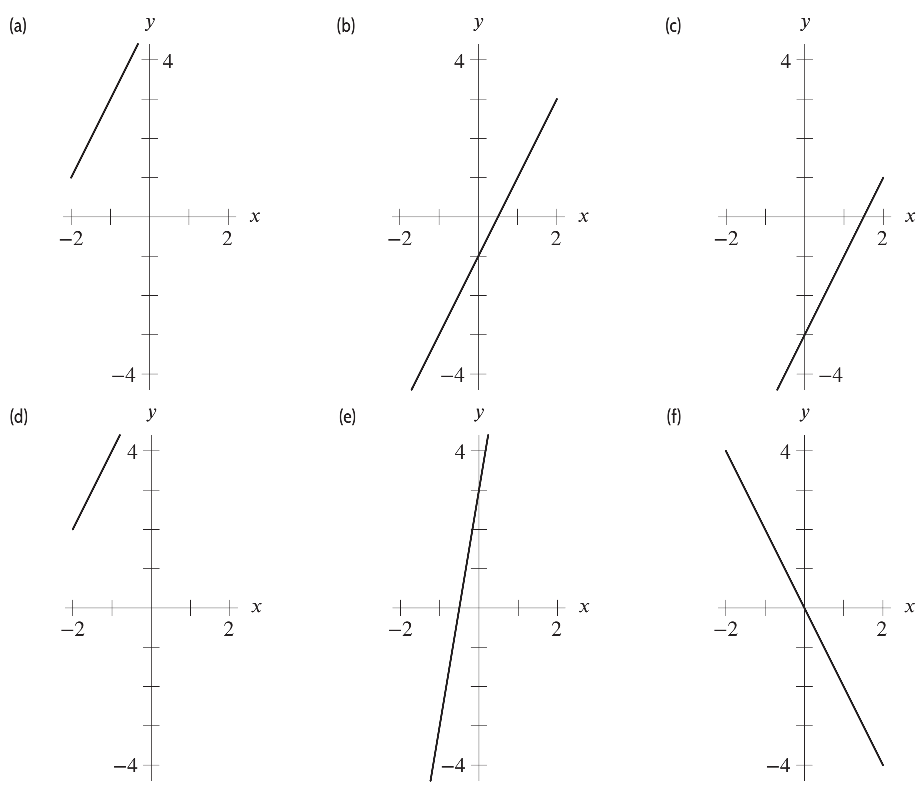
1.



2.



3.



4. This graph is the graph of $m(t)$ shifted upward by two units. See Figure 1.22.

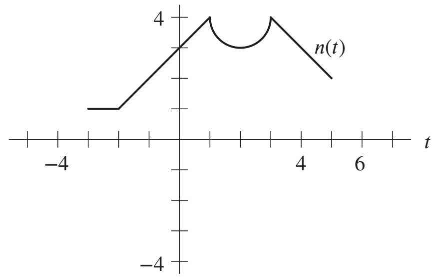


Figure 1.22

5. This graph is the graph of $m(t)$ shifted to the right by one unit. See Figure 1.23.

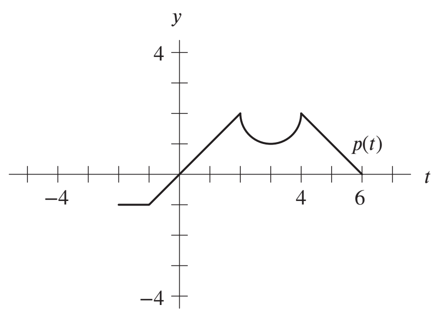


Figure 1.23

6. This graph is the graph of $m(t)$ shifted to the left by 1.5 units. See Figure 1.24.

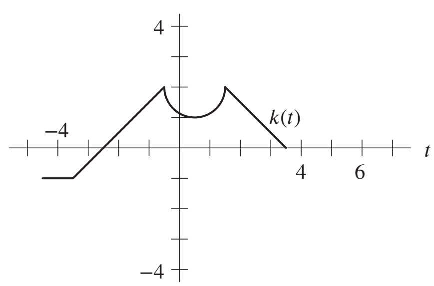


Figure 1.24

7. This graph is the graph of $m(t)$ shifted to the right by 0.5 units and downward by 2.5 units. See Figure 1.25.

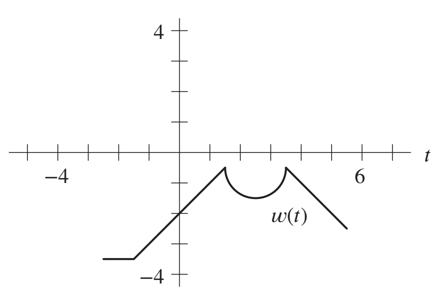


Figure 1.25

8. Figure 1.26 shows the appropriate graphs. Note that asymptotes are shown as dashed lines and x- or y-intercepts are shown as filled circles.

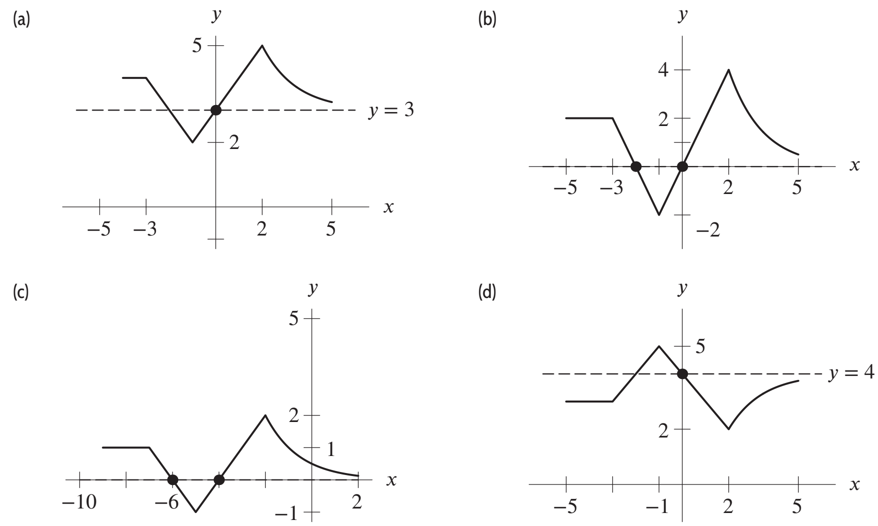


Figure 1.26

9. (a) $f(g(1)) = f(1+1) = f(2) = 2^2 = 4$
(b) $g(f(1)) = g(1^2) = g(1) = 1+1 = 2$
(c) $f(g(x)) = f(x+1) = (x+1)^2$
(d) $g(f(x)) = g(x^2) = x^2+1$
(e) $f(t)g(t) = t^2(t+1)$

10. (a) $f(g(1)) = f(1^2) = f(1) = \sqrt{1+4} = \sqrt{5}$
(b) $g(f(1)) = g(\sqrt{1+4}) = g(\sqrt{5}) = (\sqrt{5})^2 = 5$
(c) $f(g(x)) = f(x^2) = \sqrt{x^2+4}$
(d) $g(f(x)) = g(\sqrt{x+4}) = (\sqrt{x+4})^2 = x+4$
(e) $f(t)g(t) = (\sqrt{t+4})t^2 = t^2\sqrt{t+4}$

11. (a) $f(g(1)) = f(1^2) = f(1) = e^1 = e$
(b) $g(f(1)) = g(e^1) = g(e) = e^2$
(c) $f(g(x)) = f(x^2) = e^{x^2}$
(d) $g(f(x)) = g(e^x) = (e^x)^2 = e^{2x}$
(e) $f(t)g(t) = e^t t^2$

12. (a) $f(g(1)) = f(3 \cdot 1 + 4) = f(7) = \dfrac{1}{7}$
(b) $g(f(1)) = g(1/1) = g(1) = 7$
(c) $f(g(x)) = f(3x+4) = \dfrac{1}{3x+4}$
(d) $g(f(x)) = g\left(\dfrac{1}{x}\right) = 3\left(\dfrac{1}{x}\right) + 4 = \dfrac{3}{x} + 4$
(e) $f(t)g(t) = \dfrac{1}{t}(3t+4) = 3 + \dfrac{4}{t}$

13. (a) $f(t+1) = (t+1)^2 + 1 = t^2 + 2t + 1 + 1 = t^2 + 2t + 2$.
(b) $f(t^2+1) = (t^2+1)^2 + 1 = t^4 + 2t^2 + 1 + 1 = t^4 + 2t^2 + 2$.
(c) $f(2) = 2^2 + 1 = 5$.
(d) $2f(t) = 2(t^2+1) = 2t^2 + 2$.
(e) $(f(t))^2 + 1 = \left(t^2+1\right)^2 + 1 = t^4 + 2t^2 + 1 + 1 = t^4 + 2t^2 + 2$.

14. (a) $g(2+h) = (2+h)^2 + 2(2+h) + 3 = 4 + 4h + h^2 + 4 + 2h + 3 = h^2 + 6h + 11$.
(b) $g(2) = 2^2 + 2(2) + 3 = 4 + 4 + 3 = 11$, which agrees with what we get by substituting $h = 0$ into (a).
(c) $g(2+h) - g(2) = (h^2 + 6h + 11) - (11) = h^2 + 6h$.

15. $m(z+1) - m(z) = (z+1)^2 - z^2 = 2z + 1$.

16. $m(z+h)-m(z)=(z+h)^2-z^2=2zh+h^2$.

17. $m(z)-m(z-h)=z^2-(z-h)^2=2zh-h^2$.

18. $m(z+h)-m(z-h)=(z+h)^2-(z-h)^2=z^2+2hz+h^2-(z^2-2hz+h^2)=4hz$.

19.
$$f(-x)=(-x)^6+(-x)^3+1=x^6-x^3+1.$$
Since $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$, this function is neither even nor odd.

20.
$$f(-x)=(-x)^3+(-x)^2+(-x)=-x^3+x^2-x.$$
Since $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$, this function is neither even nor odd.

21. Since
$$f(-x)=(-x)^4-(-x)^2+3=x^4-x^2+3=f(x),$$
we see f is even

22. Since
$$f(-x)=(-x)^3+1=-x^3+1,$$
we see $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$, so f is neither even nor odd

23. Since
$$f(-x)=2(-x)=-2x=-f(x),$$
we see f is odd.

24. Since
$$f(-x)=e^{(-x)^2-1}=e^{x^2-1}=f(x),$$
we see f is even.

25. Since
$$f(-x)=(-x)((-x)^2-1)=-x(x^2-1)=-f(x),$$
we see f is odd

26. Since
$$f(-x)=e^{-x}+x,$$
we see $f(-x)\neq f(x)$ and $f(-x)\neq -f(x)$, so f is neither even nor odd

27. The function is not invertible since there are many horizontal lines which hit the function twice.

28. The function is not invertible since there are horizontal lines which hit the function more than once.

29. Since a horizontal line cuts the graph of $f(x)=x^2+3x+2$ two times, f is not invertible. See Figure 1.27.

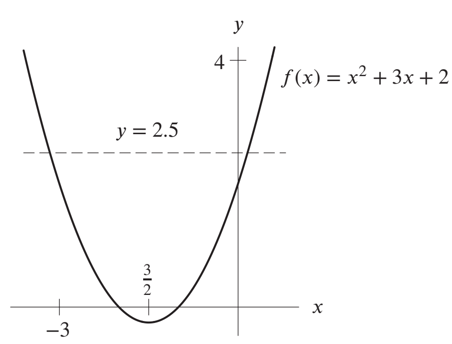


Figure 1.27

30. Since a horizontal line cuts the graph of $f(x)=x^3-5x+10$ three times, f is not invertible. See Figure 1.28.

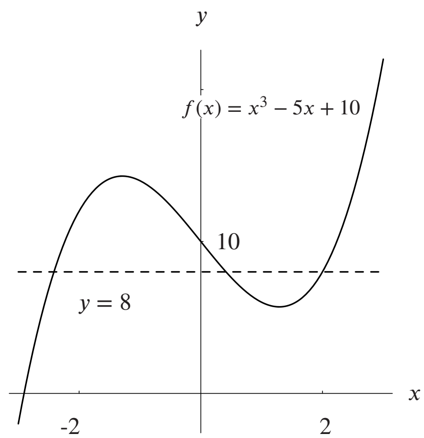


Figure 1.28

31. Since any horizontal line cuts the graph once, f is invertible. See Figure 1.29.

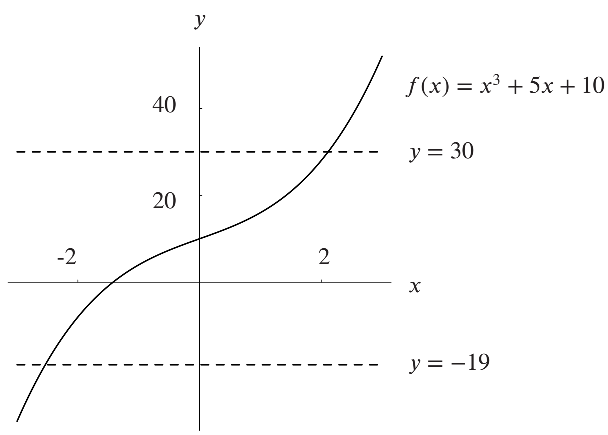


Figure 1.29

32. (a) $f(25)$ is q corresponding to $p = 25$, or, in other words, the number of items sold when the price is 25.
(b) $f^{-1}(30)$ is p corresponding to $q = 30$, or the price at which 30 units will be sold.

33. (a) $f(10{,}000)$ represents the value of C corresponding to $A = 10{,}000$, or in other words the cost of building a 10,000 square-foot store.
(b) $f^{-1}(20{,}000)$ represents the value of A corresponding to $C = 20{,}000$, or the area in square feet of a store which would cost \$20,000 to build.

34. $f^{-1}(75)$ is the length of the column of mercury in the thermometer when the temperature is 75°F.

35. (a) The equation is $y = 2x^2 + 1$. Note that its graph is narrower than the graph of $y = x^2$ which appears in gray. See Figure 1.30.

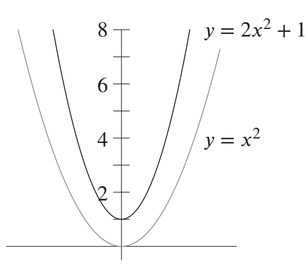


Figure 1.30

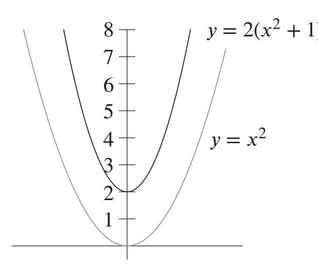


Figure 1.31

(b) $y = 2(x^2 + 1)$ moves the graph up one unit and *then* stretches it by a factor of two. See Figure 1.31.
(c) No, the graphs are not the same. Since $2(x^2 + 1) = (2x^2 + 1) + 1$, the second graph is always one unit higher than the first.

Problems

36. Since $Q = S - Se^{-kt}$, the graph of Q is the reflection of y about the t-axis moved up by S units.

37. This looks like a shift of the graph $y = -x^2$. The graph is shifted to the left 1 unit and up 3 units, so a possible formula is $y = -(x + 1)^2 + 3$.

38. This looks like a shift of the graph $y = x^3$. The graph is shifted to the right 2 units and down 1 unit, so a possible formula is $y = (x - 2)^3 - 1$.

39. We have $v(10) = 65$ but the graph of u only enables us to evaluate $u(x)$ for $0 \le x \le 50$. There is not enough information to evaluate $u(v(10))$.

40. We have approximately $v(40) = 15$ and $u(15) = 18$ so $u(v(40)) = 18$.

41. We have approximately $u(10) = 13$ and $v(13) = 60$ so $v(u(10)) = 60$.

42. We have $u(40) = 60$ but the graph of v only enables us to evaluate $v(x)$ for $0 \le x \le 50$. There is not enough information to evaluate $v(u(40))$.

43. $f(g(1)) = f(2) \approx 0.4$.

44. $g(f(2)) \approx g(0.4) \approx 1.1$.

45. $f(f(1)) \approx f(-0.4) \approx -0.9$.

46. Computing $f(g(x))$ as in Problem 43, we get Table 1.2. From it we graph $f(g(x))$ in Figure 1.32.

Table 1.2

x	$g(x)$	$f(g(x))$
−3	0.6	−0.5
−2.5	−1.1	−1.3
−2	−1.9	−1.2
−1.5	−1.9	−1.2
−1	−1.4	−1.3
−0.5	−0.5	−1
0	0.5	−0.6
0.5	1.4	−0.2
1	2	0.4
1.5	2.2	0.5
2	1.6	0
2.5	0.1	−0.7
3	−2.5	0.1

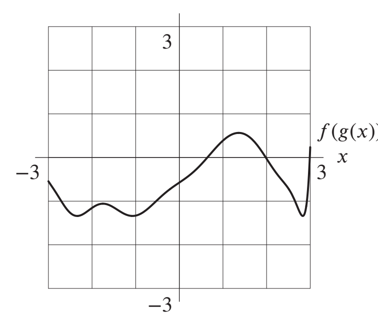


Figure 1.32

47. Using the same way to compute $g(f(x))$ as in Problem 44, we get Table 1.3. Then we can plot the graph of $g(f(x))$ in Figure 1.33.

Table 1.3

x	$f(x)$	$g(f(x))$
−3	3	−2.6
−2.5	0.1	0.8
−2	−1	−1.4
−1.5	−1.3	−1.8
−1	−1.2	−1.7
−0.5	−1	−1.4
0	−0.8	−1
0.5	−0.6	−0.6
1	−0.4	−0.3
1.5	−0.1	0.3
2	0.3	1.1
2.5	0.9	2
3	1.6	2.2

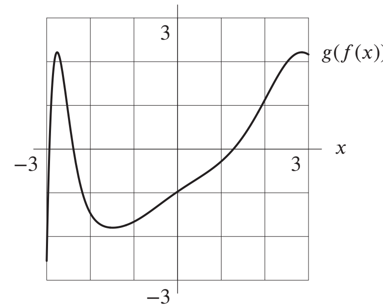


Figure 1.33

48. Using the same way to compute $f(f(x))$ as in Problem 45, we get Table 1.4. Then we can plot the graph of $f(f(x))$ in Figure 1.34.

Table 1.4

x	$f(x)$	$f(f(x))$
−3	3	1.6
−2.5	0.1	−0.7
−2	−1	−1.2
−1.5	−1.3	−1.3
−1	−1.2	−1.3
−0.5	−1	−1.2
0	−0.8	−1.1
0.5	−0.6	−1
1	−0.4	−0.9
1.5	−0.1	−0.8
2	0.3	−0.6
2.5	0.9	−0.4
3	1.6	0

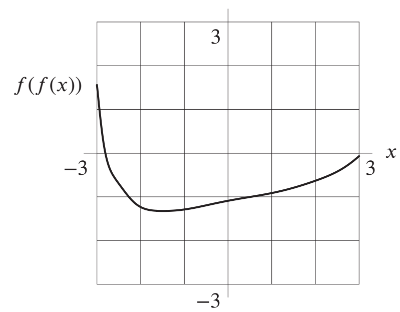


Figure 1.34

49. $f(x) = x^3, \quad g(x) = x + 1.$

50. $f(x) = x + 1, \quad g(x) = x^3.$

51. $f(x) = \sqrt{x}, \quad g(x) = x^2 + 4$

52. $f(x) = e^x, \quad g(x) = 2x$

53. The tree has $B = y - 1$ branches on average and each branch has $n = 2B^2 - B = 2(y-1)^2 - (y-1)$ leaves on average. Therefore

$$\text{Average number of leaves } = Bn = (y-1)(2(y-1)^2 - (y-1)) = 2(y-1)^3 - (y-1)^2.$$

54. The volume, V, of the balloon is $V = \frac{4}{3}\pi r^3$. When $t = 3$, the radius is 10 cm. The volume is then

$$V = \frac{4}{3}\pi(10^3) = \frac{4000\pi}{3} \text{ cm}^3.$$

55. Since f is even, we know that $f(x) = x$. Since g is odd, we know that $g(x) = -x$. Since $h(x) = g(f(x)$ and $g(x)$ is odd, we know that $h(x) = -f(x)$, and since $f(x) = x$, we know that $h(x) = -x$. Using this, we fill in Table 1.5.

Table 1.5

x	$f(x)$	$g(x)$	$h(x)$
−3	0	0	0
−2	2	2	−2
−1	2	2	−2
0	0	0	0
1	2	−2	−2
2	2	−2	−2
3	0	0	0

56. Values of f^{-1} are as follows

x	3	−7	19	4	178	2	1
$f^{-1}(x)$	1	2	3	4	5	6	7

The domain of f^{-1} is the set consisting of the integers $\{3, -7, 19, 4, 178, 2, 1\}$.

57. (a) We find $f^{-1}(2)$ by finding the x value corresponding to $f(x) = 2$. Looking at the graph, we see that $f^{-1}(2) = -1$.
(b) We construct the graph of $f^{-1}(x)$ by reflecting the graph of $f(x)$ over the line $y = x$. The graphs of $f^{-1}(x)$ and $f(x)$ are shown together in Figure 1.35.

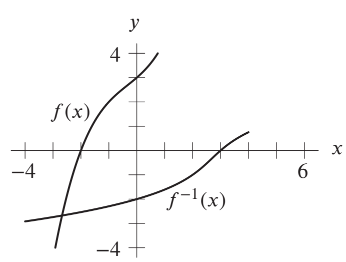


Figure 1.35

58. f is an increasing function since the amount of fuel used increases as flight time increases. Therefore f is invertible.

59. Not invertible. Given a certain number of customers, say $f(t) = 1500$, there could be many times, t, during the day at which that many people were in the store. So we don't know which time instant is the right one.

60. Probably not invertible. Since your calculus class probably has less than 363 students, there will be at least two days in the year, say a and b, with $f(a) = f(b) = 0$. Hence we don't know what to choose for $f^{-1}(0)$.

61. Not invertible, since it costs the same to mail a 50-gram letter as it does to mail a 51-gram letter.

62. The carbon footprint, in kilograms of GHGs, of a water bottle that has traveled 150 km from its production source.

63. The carbon footprint of a water bottle that travels 8700 km from its production source is 250 grams of GHGs.

64. The distance from its production source, in km, traveled by a water bottle whose carbon footprint is 1.1 kilograms of GHGs.

65. The combined carbon footprint, in kilograms of GHGs, of a water bottle that has traveled 150 km from its production source and one that has remained at the source.

66. The average per-kilometer increase in the carbon footprint of a water bottle, when the bottle travels between 150 and 1500 km from its production source.

67. The volume of the balloon t minutes after inflation began is: $g(f(t))$ ft^3.

68. The volume of the balloon if its radius were twice as big is: $g(2r)$ ft^3.

69. The time elapsed is: $f^{-1}(30)$ min.

70. The time elapsed is: $f^{-1}(g^{-1}(10{,}000))$ min.

71. (a) The function f tells us C in terms of q. To get its inverse, we want q in terms of C, which we find by solving for q:

$$\begin{aligned} C &= 100 + 2q, \\ C - 100 &= 2q, \\ q &= \frac{C - 100}{2} = f^{-1}(C). \end{aligned}$$

(b) The inverse function tells us the number of articles that can be produced for a given cost.

72. (a) Yes, f is invertible, since f is increasing everywhere.

(b) The number $f^{-1}(400)$ is the year in which 400 million motor vehicles were registered in the world. From the picture, we see that $f^{-1}(400)$ is around 1979.

(c) Since the graph of f^{-1} is the reflection of the graph of f over the line $y = x$, we get Figure 1.36.

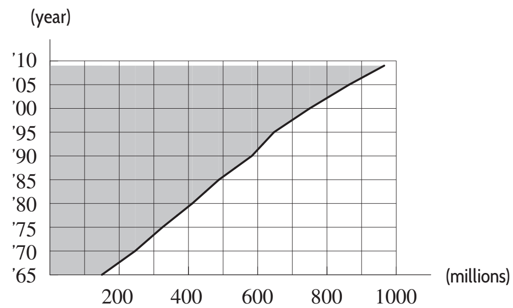


Figure 1.36: Graph of f^{-1}

73. (a) The graph shows that $f(15)$ is approximately 48. So, the place to find find 15 million-year-old rock is about 48 meters below the Atlantic sea floor.

(b) Since f is increasing (not decreasing, since the depth axis is reversed!), f is invertible. To confirm, notice that the graph of f is cut by a horizontal line at most once.

(c) Look at where the horizontal line through 120 intersects the graph of f and read downward: $f^{-1}(120)$ is about 35. In practical terms, this means that at a depth of 120 meters down, the rock is 35 million years old.

(d) First, we standardize the graph of f so that time and depth are increasing from left to right and bottom to top. Points (t, d) on the graph of f correspond to points (d, t) on the graph of f^{-1}. We can graph f^{-1} by taking points from the original graph of f, reversing their coordinates, and connecting them. This amounts to interchanging the t and d axes, thereby reflecting the graph of f about the line bisecting the 90° angle at the origin. Figure 1.37 is the graph of f^{-1}. (Note that we cannot find the graph of f^{-1} by flipping the graph of f about the line $t = d$ in because t and d have different scales in this instance.)

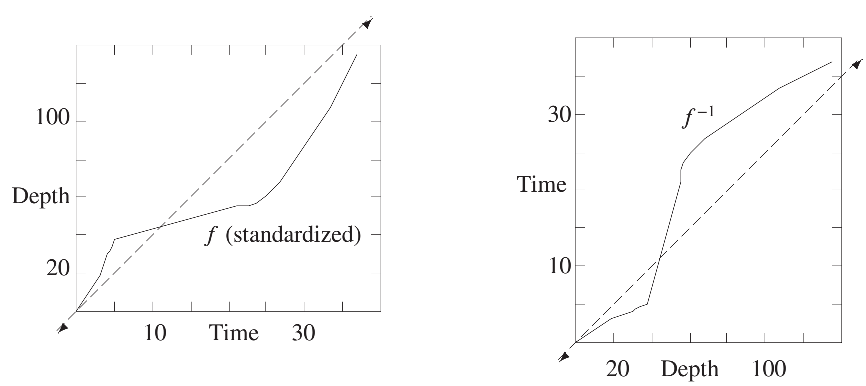


Figure 1.37: Graph of f, reflected to give that of f^{-1}

74. (a) For positive x, we have x step$(x) = x$ and for negative x we have x step$(x) = 0$. The formula describes the ramp function.
(b) For positive x, we have step(x) − step$(-x) = 1 - 0 = 1$, and for negative x we have step(x) − step$(-x) = 0 - 1 = -1$. The formula describes the sign function.
(c) For positive x, we have x step$(x) - x$ step$(-x) = x - 0 = x$, and for negative x we have x step$(x) - x$ step$(-x) = 0 - x = -x$. The formula describes the absolute value function.

Strengthen Your Understanding

75. The graph of $f(x) = -(x+1)^3$ is the graph of $g(x) = -x^3$ shifted left by 1 unit.

76. The graphs of inverse functions are symmetric with respect to the line $y = x$. Graphing these two functions for $x > 0$, we see that they are both concave up on the first quadrant, and hence not symmetric with respect to the line $y = x$.

77. Since $f(g(x)) = 3(-3x - 5) + 5 = -9x - 10$, we see that f and g are not inverse functions.

78. Two inverse functions must equal the identity function when composed with each other. This is not true in this case:

$$f(f(x)) = f(e^x) = e^{e^x} = (e^e)^x \approx (15.154)^x \neq x.$$

79. While $y = 1/x$ is sometimes referred to as the *multiplicative* inverse of x, the inverse of f is $f^{-1}(x) = x$.

80. One possible answer is $g(x) = 3 + x$. (There are many answers.)

81. One possibility is $f(x) = x^2 + 2$.

82. Let $f(x) = 3x$, then $f^{-1}(x) = x/3$. Then for $x > 0$, we have $f(x) > f^{-1}(x)$.

83. We have

$$g(x) = f(x+2)$$

because the graph of g is obtained by moving the graph of f to the left by 2 units. We also have

$$g(x) = f(x) + 3$$

because the graph of g is obtained by moving the graph of f up by 3 units. Thus, we have $f(x+2) = f(x)+3$. The graph of f climbs 3 units whenever x increases by 2. The simplest choice for f is a linear function of slope $3/2$, for example $f(x) = 1.5x$, so $g(x) = 1.5x + 3$.

84. True. Using properties of exponents we see that:

$$f(g(x)) = f(3^x) = 2^{3^x} = \left(2^3\right)^x = 8^x.$$

So, we see $f(g(x)) = 8^x$ is exponential. Similarly, we can check $g(f(x)) = 9^x$ is exponential. In general, the composition of two exponential functions of form $y = Ca^x$ is exponential.

85. True. The graph of $y = 10^x$ is moved horizontally by h units if we replace x by $x-h$ for some number h. Writing $100 = 10^2$, we have $f(x) = 100(10^x) = 10^2 \cdot 10^x = 10^{x+2}$. The graph of $f(x) = 10^{x+2}$ is the graph of $g(x) = 10^x$ shifted two units to the left.

86. True. If f is increasing then its reflection about the line $y = x$ is also increasing. An example is shown in Figure 1.38. The statement is true.

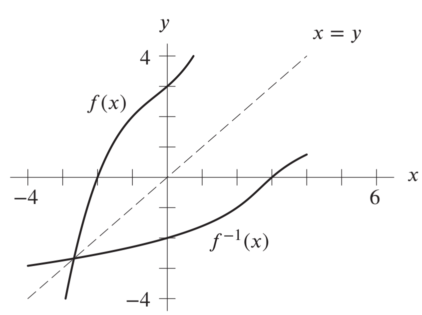


Figure 1.38

87. True. If $f(x)$ is even, we have $f(x) = f(-x)$ for all x. For example, $f(-2) = f(2)$. This means that the graph of $f(x)$ intersects the horizontal line $y = f(2)$ at two points, $x = 2$ and $x = -2$. Thus, f has no inverse function.

88. False. For example, $f(x) = x$ and $g(x) = x^3$ are both odd. Their inverses are $f^{-1}(x) = x$ and $g^{-1}(x) = x^{1/3}$.

89. False. For $x < 0$, as x increases, x^2 decreases, so e^{-x^2} increases.

90. True. We have $g(-x) = g(x)$ since g is even, and therefore $f(g(-x)) = f(g(x))$.

91. False. A counterexample is given by $f(x) = x^2$ and $g(x) = x + 1$. The function $f(g(x)) = (x + 1)^2$ is not even because $f(g(1)) = 4$ and $f(g(-1)) = 0 \neq 4$.

92. True. The constant function $f(x) = 0$ is the only function that is both even and odd. This follows, since if f is both even and odd, then, for all x, $f(-x) = f(x)$ (if f is even) and $f(-x) = -f(x)$ (if f is odd). Thus, for all x, $f(x) = -f(x)$ i.e. $f(x) = 0$, for all x. So $f(x) = 0$ is both even and odd and is the only such function.

93. True. A linear function can be written as $y = b + mx$. Hence, writing $f(x) = c + nx$ and $g(x) = d + px$ and using algebra, we see that:

$$f(g(x)) = f(d + px) = c + n(d + px) = c + nd + npx.$$

So, setting $b = c + nd$ and $m = np$, we see that $f(g(x))$ is of the form $b + mx$. Similarly, we can check $g(f(x))$ is of this form.

94. Let $f(x) = x$ and $g(x) = -2x$. Then $f(x) + g(x) = -x$, which is decreasing. Note f is increasing since it has positive slope, and g is decreasing since it has negative slope.

95. This is impossible. If $a < b$, then $f(a) < f(b)$, since f is increasing, and $g(a) > g(b)$, since g is decreasing, so $-g(a) < -g(b)$. Therefore, if $a < b$, then $f(a) - g(a) < f(b) - g(b)$, which means that $f(x) + g(x)$ is increasing.

96. Let $f(x) = e^x$ and let $g(x) = e^{-2x}$. Note f is increasing since it is an exponential growth function, and g is decreasing since it is an exponential decay function. Then $f(x)g(x) = e^{-x}$, which is decreasing.

97. This is impossible. As x increases, $g(x)$ decreases. As $g(x)$ decreases, so does $f(g(x))$ because f is increasing (an increasing function increases as its variable increases, so it decreases as its variable decreases).

Solutions for Section 1.4

Exercises

1. Using the identity $e^{\ln x} = x$, we have $e^{\ln(1/2)} = \frac{1}{2}$.

2. Using the identity $10^{\log x} = x$, we have

$$10^{\log(AB)} = AB$$

3. Using the identity $e^{\ln x} = x$, we have $5A^2$.

4. Using the identity $\ln(e^x) = x$, we have 2AB.

5. Using the rules for ln, we have

$$\begin{aligned}\ln\left(\frac{1}{e}\right) + \ln AB &= \ln 1 - \ln e + \ln A + \ln B\\ &= 0 - 1 + \ln A + \ln B\\ &= -1 + \ln A + \ln B.\end{aligned}$$

6. Using the rules for ln, we have $2A + 3e \ln B$.

7. Taking logs of both sides

$$\begin{aligned}\log 3^x &= x \log 3 = \log 11\\ x &= \frac{\log 11}{\log 3} = 2.2.\end{aligned}$$

8. Taking logs of both sides

$$\begin{aligned} \log 17^x &= \log 2 \\ x\log 17 &= \log 2 \\ x &= \frac{\log 2}{\log 17} \approx 0.24. \end{aligned}$$

9. Isolating the exponential term

$$\begin{aligned} 20 &= 50(1.04)^x \\ \frac{20}{50} &= (1.04)^x. \end{aligned}$$

Taking logs of both sides

$$\begin{aligned} \log\frac{2}{5} &= \log(1.04)^x \\ \log\frac{2}{5} &= x\log(1.04) \\ x &= \frac{\log(2/5)}{\log(1.04)} = -23.4. \end{aligned}$$

10.

$$\begin{aligned} \frac{4}{7} &= \frac{5^x}{3^x} \\ \frac{4}{7} &= \left(\frac{5}{3}\right)^x \end{aligned}$$

Taking logs of both sides

$$\begin{aligned} \log\left(\frac{4}{7}\right) &= x\log\left(\frac{5}{3}\right) \\ x &= \frac{\log(4/7)}{\log(5/3)} \approx -1.1. \end{aligned}$$

11. To solve for x, we first divide both sides by 5 and then take the natural logarithm of both sides.

$$\begin{aligned} \frac{7}{5} &= e^{0.2x} \\ \ln(7/5) &= 0.2x \\ x &= \frac{\ln(7/5)}{0.2} \approx 1.68. \end{aligned}$$

12.
$$\begin{aligned} \ln(2^x) &= \ln(e^{x+1}) \\ x\ln 2 &= (x+1)\ln e \\ x\ln 2 &= x+1 \\ 0.693x &= x+1 \\ x &= \frac{1}{0.693-1} \approx -3.26 \end{aligned}$$

13. To solve for x, we first divide both sides by 600 and then take the natural logarithm of both sides.

$$\begin{aligned} \frac{50}{600} &= e^{-0.4x} \\ \ln(50/600) &= -0.4x \\ x &= \frac{\ln(50/600)}{-0.4} \approx 6.212. \end{aligned}$$

14.
$$\begin{aligned}
\ln(2e^{3x}) &= \ln(4e^{5x})\\
\ln 2 + \ln(e^{3x}) &= \ln 4 + \ln(e^{5x})\\
0.693 + 3x &= 1.386 + 5x\\
x &= -0.347
\end{aligned}$$

15. Using the rules for ln, we get

$$\begin{aligned}
\ln 7^{x+2} &= \ln e^{17x}\\
(x+2)\ln 7 &= 17x\\
x(\ln 7 - 17) &= -2\ln 7\\
x &= \frac{-2\ln 7}{\ln 7 - 17} \approx 0.26.
\end{aligned}$$

16.
$$\begin{aligned}
\ln(10^{x+3}) &= \ln(5e^{7-x})\\
(x+3)\ln 10 &= \ln 5 + (7-x)\ln e\\
2.303(x+3) &= 1.609 + (7-x)\\
3.303x &= 1.609 + 7 - 2.303(3)\\
x &= 0.515
\end{aligned}$$

17. Using the rules for ln, we have

$$\begin{aligned}
2x - 1 &= x^2\\
x^2 - 2x + 1 &= 0\\
(x-1)^2 &= 0\\
x &= 1.
\end{aligned}$$

18.
$$\begin{aligned}
4e^{2x-3} &= e + 5\\
\ln 4 + \ln(e^{2x-3}) &= \ln(e+5)\\
1.3863 + 2x - 3 &= 2.0436\\
x &= 1.839.
\end{aligned}$$

19. $t = \dfrac{\log a}{\log b}$.

20. $t = \dfrac{\log\left(\frac{P}{P_0}\right)}{\log a} = \dfrac{\log P - \log P_0}{\log a}$.

21. Taking logs of both sides yields

$$nt = \frac{\log\left(\frac{Q}{Q_0}\right)}{\log a}.$$

Hence

$$t = \frac{\log\left(\frac{Q}{Q_0}\right)}{n\log a} = \frac{\log Q - \log Q_0}{n\log a}.$$

22. Collecting similar terms yields

$$\left(\frac{a}{b}\right)^t = \frac{Q_0}{P_0}.$$

Hence

$$t = \frac{\log\left(\frac{Q_0}{P_0}\right)}{\log\left(\frac{a}{b}\right)}.$$

23. $t = \ln \dfrac{a}{b}$.

24. $\ln \dfrac{P}{P_0} = kt$, so $t = \dfrac{\ln \frac{P}{P_0}}{k}$.

25. Since we want $(1.5)^t = e^{kt} = (e^k)^t$, so $1.5 = e^k$, and $k = \ln 1.5 = 0.4055$. Thus, $P = 15e^{0.4055t}$. Since 0.4055 is positive, this is exponential growth.

26. We want $1.7^t = e^{kt}$ so $1.7 = e^k$ and $k = \ln 1.7 = 0.5306$. Thus $P = 10e^{0.5306t}$.

27. We want $0.9^t = e^{kt}$ so $0.9 = e^k$ and $k = \ln 0.9 = -0.1054$. Thus $P = 174e^{-0.1054t}$.

28. Since we want $(0.55)^t = e^{kt} = (e^k)^t$, so $0.55 = e^k$, and $k = \ln 0.55 = -0.5978$. Thus $P = 4e^{-0.5978t}$. Since -0.5978 is negative, this represents exponential decay.

29. If $p(t) = (1.04)^t$, then, for p^{-1} the inverse of p, we should have

$$\begin{aligned}
(1.04)^{p^{-1}(t)} &= t,\\
p^{-1}(t)\log(1.04) &= \log t,\\
p^{-1}(t) &= \frac{\log t}{\log(1.04)} \approx 58.708 \log t.
\end{aligned}$$

30. Since f is increasing, f has an inverse. To find the inverse of $f(t) = 50e^{0.1t}$, we replace t with $f^{-1}(t)$, and, since $f(f^{-1}(t)) = t$, we have

$$t = 50e^{0.1f^{-1}(t)}.$$

We then solve for $f^{-1}(t)$:

$$\begin{aligned}
t &= 50e^{0.1f^{-1}(t)}\\
\frac{t}{50} &= e^{0.1f^{-1}(t)}\\
\ln\left(\frac{t}{50}\right) &= 0.1f^{-1}(t)\\
f^{-1}(t) &= \frac{1}{0.1}\ln\left(\frac{t}{50}\right) = 10\ln\left(\frac{t}{50}\right).
\end{aligned}$$

31. Using $f(f^{-1}(t)) = t$, we see

$$f(f^{-1}(t)) = 1 + \ln f^{-1}(t) = t.$$

So

$$\begin{aligned}
\ln f^{-1}(t) &= t - 1\\
f^{-1}(t) &= e^{t-1}.
\end{aligned}$$

Problems

32. Since $y(0) = Ce^0 = C$ we have that $C = 2$. Similarly, substituting $x = 1$ gives $y(1) = 2e^{\alpha}$ so

$$2e^{\alpha} = 1.$$

Rearranging gives $e^{\alpha} = 1/2$. Taking logarithms we get $\alpha = \ln(1/2) = -\ln 2 = -0.693$. Finally,

$$y(2) = 2e^{2(-\ln 2)} = 2e^{-2\ln 2} = \frac{1}{2}.$$

33. At the doubling time, $t = 10$, we have $p = 2p_0$. Thus

$$\begin{aligned}
p_0e^{10k} &= 2p_0\\
e^{10k} &= 2\\
10k &= \ln 2\\
k &= \frac{1}{10}\ln 2 = 0.0693.
\end{aligned}$$

The function $p = p_0e^{0.0693t}$ has doubling time equal to 10.

34. At the doubling time, $t = 0.4$, the we have $p = 2p_0$. Thus

$$\begin{aligned} p_0 e^{0.4k} &= 2p_0 \\ e^{0.4k} &= 2 \\ 0.4k &= \ln 2 \\ k &= \frac{1}{0.4}\ln 2 = 1.733. \end{aligned}$$

The function $p = p_0 e^{1.733t}$ has doubling time equal to 0.4.

35. We can solve for the growth rate k of the bacteria using the formula $P = P_0 e^{kt}$:

$$\begin{aligned} 1500 &= 500e^{k(2)} \\ k &= \frac{\ln(1500/500)}{2}. \end{aligned}$$

Knowing the growth rate, we can find the population P at time $t = 6$:

$$\begin{aligned} P &= 500e^{(\frac{\ln 3}{2})6} \\ &\approx 13{,}500 \text{ bacteria.} \end{aligned}$$

36. In ten years, the substance has decayed to 40% of its original mass. In another ten years, it will decay by an additional factor of 40%, so the amount remaining after 20 years will be $100 \cdot 40\% \cdot 40\% = 16$ kg.

37. (a) Assuming the US population grows exponentially, we have population $P(t) = 281.4e^{kt}$ at time t years after 2000. Using the 2013 population, we have

$$\begin{aligned} 316.1 &= 281.4e^{13k} \\ k &= \frac{\ln(316.1/281.4)}{13} = 0.00894. \end{aligned}$$

We want to find the time t in which

$$\begin{aligned} 350 &= 281.4e^{0.00894t} \\ t &= \frac{\ln(350/281.4)}{0.00926} = 24.4 \text{ years.} \end{aligned}$$

This model predicts the population to go over 350 million 24.4 years after 2000, in the year 2024.

(b) Evaluate $P = 281.4e^{0.00894t}$ for $t = 20$ to find $P = 336.49$ million people.

38. The population has increased by a factor of $48{,}000{,}000/40{,}000{,}000 = 1.2$ in 10 years. Thus we have the formula

$$P = 40{,}000{,}000(1.2)^{t/10},$$

and $t/10$ gives the number of 10-year periods that have passed since 2005.

In 2005, $t/10 = 0$, so we have $P = 40{,}000{,}000$.
In 2015, $t/10 = 1$, so $P = 40{,}000{,}000(1.2) = 48{,}000{,}000$.
In 2020, $t/10 = 1.5$, so $P = 40{,}000{,}000(1.2)^{1.5} \approx 52{,}581{,}000$.
To find the doubling time, solve $80{,}000{,}000 = 40{,}000{,}000(1.2)^{t/10}$, to get $t = 38.02$ years.

39. Let $f(t)$ be oil consumption t years after 2010. Since consumption is exponential, and $f(0) = 8.938$ million barrels per day, we have

$$f(t) = 8.938e^{kt}.$$

In 2013, when $t = 3$, consumption was 10.480, so

$$\begin{aligned} f(3) &= 10.480 \\ 8.938e^{3t} &= 10.480 \\ 3t &= \ln\left(\frac{10.480}{8.938}\right) \\ k &= 0.05305. \end{aligned}$$

Our model predicts Chinese oil consumption in 2025, when $t = 15$, to be

$$f(15) = 8.938e^{0.05305(15)} = 19.807 \text{ million barrels per day.}$$

40. If C_0 is the concentration of NO_2 on the road, then the concentration x meters from the road is

$$C = C_0 e^{-0.0254x}.$$

We want to find the value of x making $C = C_0/2$, that is,

$$C_0 e^{-0.0254x} = \frac{C_0}{2}.$$

Dividing by C_0 and then taking natural logs yields

$$\ln\left(e^{-0.254x}\right) = -0.0254x = \ln\left(\frac{1}{2}\right) = -0.6931,$$

so

$$x = 27 \text{ meters}.$$

At 27 meters from the road the concentration of NO_2 in the air is half the concentration on the road.

41. (a) Since the percent increase in deaths during a year is constant for constant increase in pollution, the number of deaths per year is an exponential function of the quantity of pollution. If Q_0 is the number of deaths per year without pollution, then the number of deaths per year, Q, when the quantity of pollution is x micrograms per cubic meter of air is

$$Q = Q_0(1.0033)^x.$$

(b) We want to find the value of x making $Q = 2Q_0$, that is,

$$Q_0(1.0033)^x = 2Q_0.$$

Dividing by Q_0 and then taking natural logs yields

$$\ln((1.0033)^x) = x \ln 1.0033 = \ln 2,$$

so

$$x = \frac{\ln 2}{\ln 1.0033} = 210.391.$$

When there are 210.391 micrograms of pollutants per cubic meter of air, respiratory deaths per year are double what they would be in the absence of air pollution.

42. (a) Since there are 5 years between 2005 and 2010 we let t be the number of years since 2005 and get:

$$618{,}505 = 246{,}363e^{5r}.$$

Solving for r, we get

$$\frac{618{,}505}{246{,}363} = e^{5r}$$
$$\ln\left(\frac{618{,}505}{246{,}363}\right) = 5r$$
$$r = 0.18410.$$

Substituting $t = 1, 2, 3, 4$ into

$$246{,}363\ e^{(0.18410)t},$$

we find the remaining table values:

Year	2005	2006	2007	2008	2009	2010
Number of E85 vehicles	246,363	296,132	355,956	427,864	514,300	618,505

(b) If N is the number of E85-powered vehicles in 2004, then

$$246{,}363 = Ne^{0.18410t}$$

or

$$N = \frac{246{,}363}{e^{0.18410}} = 204{,}958 \text{ vehicles}.$$

(c) From the table, we can see that the number of E85 vehicles slightly more than doubled from 2005 to 2009, so the percent growth between these years should be slightly over 100%:

$$\begin{array}{c}\text{Percent growth from}\\ \text{2005 to 2009}\end{array} = 100\left(\frac{514{,}300}{246{,}363} - 1\right) = 1.108575 = 108.575\%.$$

43. (a) Since the initial amount of caffeine is 100 mg and the exponential decay rate is -0.17, we have $A = 100e^{-0.17t}$.
(b) See Figure 1.39. We estimate the half-life by estimating t when the caffeine is reduced by half (so $A = 50$); this occurs at approximately $t = 4$ hours.

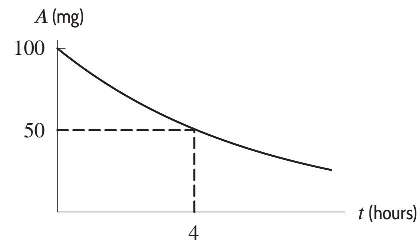


Figure 1.39

(c) We want to find the value of t when $A = 50$:

$$\begin{aligned} 50 &= 100e^{-0.17t} \\ 0.5 &= e^{-0.17t} \\ \ln 0.5 &= -0.17t \\ t &= 4.077. \end{aligned}$$

The half-life of caffeine is about 4.077 hours. This agrees with what we saw in Figure 1.39.

44. (a) The concentration decreases from 100 to 50 during the time from 1970 to 1975, a period of 5 years.
(b) The concentration decreases from 50 to 25 during the time from 1975 to 1980, a period of 5 years.
(c) The decay is exponential if the half-life of the concentration is the same beginning at any concentration level. We only checked at two levels, 100 and 50, but for those the half-lives were both 5 years. This is some evidence that the decay might be exponential.

Also, if we look at any other 5-year period, we can check that the concentration appears to decreases by half.

(d) Since the initial value of concentration is 100, an exponential model is of the form $C = 100e^{kt}$. We can evaluate k by using the value $C = 25$ at $t = 10$.

$$\begin{aligned} 25 &= 100e^{10k} \\ 0.25 &= e^{10k} \\ \ln(0.25) &= 10k \\ k &= -0.139. \end{aligned}$$

An exponential model is given by

$$C = 100e^{-0.139t}.$$

Alternative solution. Using the half-life $T = 5$ years, we have

$$C = 100 \cdot 2^{-t/T} = 100 \cdot 2^{-t/5}.$$

45. (a) The initial dose is 10 mg.
(b) Since $0.82 = 1 - 0.18$, the decay rate is 0.18, so 18% leaves the body each hour.
(c) When $t = 6$, we have $A = 10(0.82)^6 = 3.04$. The amount in the body after 6 hours is 3.04 mg.
(d) We want to find the value of t when $A = 1$. Using logarithms:

$$\begin{aligned} 1 &= 10(0.82)^t \\ 0.1 &= (0.82)^t \\ \ln(0.1) &= t\ln(0.82) \\ t &= 11.60 \text{ hours}. \end{aligned}$$

After 11.60 hours, the amount is 1 mg.

46. To find a half-life, we want to find at what t value $Q = \frac{1}{2}Q_0$. Plugging this into the equation of the decay of plutonium-240, we have

$$\begin{aligned}\frac{1}{2} &= e^{-0.00011t}\\ t &= \frac{\ln(1/2)}{-0.00011} \approx 6{,}301 \text{ years.}\end{aligned}$$

The only difference in the case of plutonium-242 is that the constant -0.00011 in the exponent is now -0.0000018. Thus, following the same procedure, the solution for t is

$$t = \frac{\ln(1/2)}{-0.0000018} \approx 385{,}081 \text{ years.}$$

47. Given the doubling time of 5 hours, we can solve for the bacteria's growth rate;

$$\begin{aligned}2P_0 &= P_0e^{k5}\\ k &= \frac{\ln 2}{5}.\end{aligned}$$

So the growth of the bacteria population is given by:

$$P = P_0e^{\ln(2)t/5}.$$

We want to find t such that

$$3P_0 = P_0e^{\ln(2)t/5}.$$

Therefore we divide both sides by P_0 and apply ln. We get

$$t = \frac{5\ln(3)}{\ln(2)} = 7.925 \text{ hours.}$$

48. (a) The pressure P at 6194 meters is given in terms of the pressure P_0 at sea level to be

$$\begin{aligned}P &= P_0e^{-0.00012h}\\ &= P_0e^{(-0.00012)6194}\\ &= P_0e^{-0.74328}\\ &\approx 0.4756P_0 \quad \text{or about 47.6\% of sea level pressure.}\end{aligned}$$

(b) At $h = 12{,}000$ meters, we have

$$\begin{aligned}P &= P_0e^{-0.00012h}\\ &= P_0e^{(-0.00012)12{,}000}\\ &= P_0e^{-1.44}\\ &\approx 0.2369P_0 \quad \text{or about 23.7\% of sea level pressure.}\end{aligned}$$

49. (a) $B(t) = B_0e^{0.067t}$
(b) $P(t) = P_0e^{0.033t}$
(c) If the initial price is \$50, then

$$\begin{aligned}B(t) &= 50e^{0.067t}\\ P(t) &= 50e^{0.033t}.\end{aligned}$$

We want the value of t such that

$$\begin{aligned}B(t) &= 2P(t)\\ 50e^{0.067t} &= 2\cdot 50e^{0.033t}\\ \frac{e^{0.067t}}{e^{0.033t}} &= e^{0.034t} = 2\\ t &= \frac{\ln 2}{0.034} = 20.387 \text{ years .}\end{aligned}$$

Thus, when $t = 20.387$ the price of the textbook was predicted to be double what it would have been had the price risen by inflation only. This occurred in the year 2000.

50. (a) We assume $f(t) = Ae^{-kt}$, where A is the initial population, so $A = 100{,}000$. When $t = 110$, there were 3200 tigers, so

$$3200 = 100{,}000e^{-k\cdot 110}$$

Solving for k gives

$$e^{-k\cdot 110} = \frac{3200}{100{,}000} = 0.0132$$
$$k = -\frac{1}{110}\ln(0.0132) = 0.0313 = 3.13\%$$

so

$$f(t) = 100{,}000e^{-0.0313t}.$$

(b) In 2000, the predicted number of tigers was

$$f(100) = 100{,}000e^{-0.0313(100)} = 4372.$$

In 2010, we know the number of tigers was 3200. The predicted percent reduction is

$$\frac{3200 - 4372}{4372} = -0.268 = -26.8\%.$$

Thus the actual decrease is larger than the predicted decrease.

51. The population of China, C, in billions, is given by

$$C = 1.355(1.0044)^t$$

where t is time measured from 2014, and the population of India, I, in billions, is given by

$$I = 1.255(1.0125)^t.$$

The two populations will be equal when $C = I$, thus, we must solve the equation:

$$1.355(1.004)^t = 1.255(1.0125)^t$$

for t, which leads to

$$\frac{1.355}{1.255} = \frac{(1.0125)^t}{(1.0044)^t} = \left(\frac{1.0125}{1.0044}\right)^t.$$

Taking logs on both sides, we get

$$t\log\frac{1.0125}{1.0044} = \log\frac{1.355}{1.255},$$

so

$$t = \frac{\log(1.355/1.255)}{\log(1.0125/1.0044)} = 9.54 \text{ years}.$$

This model predicts the population of India will exceed that of China in 2023.

52. Let A represent the revenue (in billions of dollars) at Apple t years since 2013. Since $A = 54.5$ when $t = 0$ and we want the continuous growth rate, we write $A = 54.5e^{kt}$. We use the information from 2015, that $A = 74.6$ when $t = 2$, to find k:

$$74.6 = 54.5e^{k\cdot 2}$$
$$1.37 = e^{2k}$$
$$\ln(1.37) = 2k$$
$$k = 0.157.$$

We have $A = 54.5e^{0.157t}$, which represents a continuous growth rate of 15.7% per year.

53. Let $P(t)$ be the world population in billions t years after 2010.

(a) Assuming exponential growth, we have

$$P(t) = 6.9e^{kt}.$$

In 2050, we have $t = 40$ and we expect the population then to be 9 billion, so

$$9 = 6.9e^{k\cdot 40}.$$

Solving for k, we have

$$e^{k\cdot 40} = \frac{9}{6.9}$$
$$k = \frac{1}{40}\ln\left(\frac{9}{6.9}\right) = 0.00664 = 0.664\% \text{ per year}.$$

(b) The "Day of 8 Billion" should occur when

$$8 = 6.9e^{0.00664t}.$$

Solving for t gives

$$\begin{aligned} e^{0.00664t} &= \frac{8}{6.9} \\ t &= \frac{\ln(8/6.9)}{0.00664} = 22.277 \text{ years.} \end{aligned}$$

So the "Day of 8 Billion" should be 22.277 years after the end of 2010. This is 22 years and $0.227 \cdot 365 \approx 101$ days; so 101 days into 2032. That is, April 11, 2032.

54. If r was the average yearly inflation rate, in decimals, then $\frac{1}{4}(1+r)^3 = 2{,}400{,}000$, so $r = 211.53$, i.e. $r = 21{,}153\%$.

55. If t is time in decades, then the number of vehicles, V, in millions, is given by

$$V = 246(1.155)^t.$$

For time t in decades, the number of people, P, in millions, is given by

$$P = 308.7(1.097)^t.$$

There is an average of one vehicle per person when $\frac{V}{P} = 1$, or $V = P$. Thus, we solve for t in the equation:

$$246(1.155)^t = 308.7(1.097)^t,$$

which leads to

$$\left(\frac{1.155}{1.097}\right)^t = \frac{(1.155)^t}{(1.097)^t} = \frac{308.7}{246}$$

Taking logs on both sides, we get

$$t \log \frac{1.155}{1.097} = \log \frac{308.7}{246},$$

so

$$t = \frac{\log(308.7/246)}{\log(1.155/1.097)} = 4.41 \text{ decades.}$$

This model predicts one vehicle per person in 2054

56. For an exponentially growing population, $p = p_0e^{kt}$, that doubles in n days, we find the continuous growth rate k in terms of n. Since $p = 2p_0$ when $t = n$, we have

$$\begin{aligned} 2p_0 &= p_0e^{kn} \\ k &= \frac{1}{n}\ln 2 \\ p &= p_0e^{(\ln 2)t/n}. \end{aligned}$$

Doubling time for the phytoplankton population, P, is 0.5 days, and for the foraminifera population, F, it is 5 days. If they both initially have the same population C, then after t days we have

$$\begin{aligned} P &= Ce^{(\ln 2)t/0.5} = Ce^{1.3863t} \\ F &= Ce^{(\ln 2)t/5} = Ce^{0.13863t}. \end{aligned}$$

(a) We have

$$\begin{aligned} P &= 2F \\ Ce^{1.3863t} &= 2Ce^{0.13863t} \\ e^{1.3863t} &= 2e^{0.13863t} \\ 1.3863t &= \ln 2 + 0.13863t \\ t &= 0.556 \text{ days.} \end{aligned}$$

The phytoplankton population is double the foraminifera population after 0.556 days, a little over 13 hours.

(b) We have

$$\begin{aligned} P &= 1000F \\ Ce^{1.3863t} &= 1000Ce^{0.13863t} \\ e^{1.3863t} &= 1000e^{0.13863t} \\ 1.3863t &= \ln 1000 + 0.13863t \\ t &= 5.537 \text{ days.} \end{aligned}$$

The phytoplankton population is 1000 times the foraminifera population after 5.537 days, about five and a half days.

57. We assume exponential decay and solve for k using the half-life:

$$e^{-k(5730)} = 0.5 \quad \text{so} \quad k = 1.21 \cdot 10^{-4}.$$

Now find t, the age of the painting:

$$e^{-1.21 \cdot 10^{-4} t} = 0.995, \quad \text{so} \quad t = \frac{\ln 0.995}{-1.21 \cdot 10^{-4}} = 41.43 \text{ years.}$$

Since Vermeer died in 1675, the painting is a fake.

58. (a) We find an exponential model in the form $c(t) = c_0 a^t$. Using $c(0) = 25$ and $c(1) = 21.8$, we have

$$c(t) = 25a^t$$

$$21.8 = 25a^1 = 25a.$$

Thus

$$\begin{aligned} a &= \frac{21.8}{25} = 0.872 \\ c(t) &= 25(0.872)^t. \end{aligned}$$

Alternately, we could use $c(t) = c_0 e^{kt}$. Then we have $21.8 = 25e^{k \cdot 1} = 25e^k$, giving

$$k = \ln\left(\frac{21.8}{25}\right) = \ln(0.872) = -0.136966$$

$$c(t) = 25e^{-0.136966t}.$$

(b) We find t so that $25(0.872)^t = 10$, giving

$$\begin{aligned} (0.872)^t &= \frac{10}{25} \\ t \ln(0.872) &= \ln\left(\frac{10}{25}\right) \\ t &= \frac{\ln(10/25)}{\ln(0.872)} = 6.6899 = 6.690 \text{ years.} \end{aligned}$$

Alternatively, solving $25e^{-0.136966t} = 10$ gives

$$t = -\frac{\ln(10/25)}{0.136966} = 6.6899 = 6.690 \text{ years.}$$

(c) If $D(t) = c(2t)$, then

$$D(t) = 25(0.872)^{2t} = 25\left(0.872^2\right)^t = 25(0.7604)^t.$$

Alternatively,

$$D(t) = 25e^{-0.136966(2t)} = 25e^{-0.273932t}.$$

(d) Starting three years earlier, but with t measured from the original time, we have $E(t) = c(t+3)$, so

$$E(t) = 25(0.872)^{t+3} = 25(0.872)^3(0.872)^t = 16.576(0.872)^t.$$

Alternatively,

$$E(t) = 25e^{-0.136966(t+3)} = 25e^{-0.136966(3)}e^{-0.136966t} = 16.576e^{-0.136966t}.$$

59. Let W_1 be the strength of the first earthquake and let W_2 be the strength of the second earthquake. Then

$$R_1 = \log\left(\frac{W_1}{W_0}\right) = 7.8$$

$$R_2 = \log\left(\frac{W_2}{W_0}\right) = 7.3.$$

(a) We want to know W_1/W_0, so we raise the expression for R_1 to the tenth power:

$$\frac{W_1}{W_0} = 10^{7.8} = 63{,}095{,}734 \approx 63 \text{ million.}$$

(b) We want to know W_1/W_2, so we subtract the two expressions for R_1 and R_2, giving

$$R_1 - R_2 = \log\left(\frac{W_1}{W_0}\right) - \log\left(\frac{W_2}{W_0}\right) = \log\left(\frac{W_1/W_0}{W_2/W_0}\right) = \log\left(\frac{W_1}{W_2}\right) = 7.8 - 7.3 = 0.5.$$

To find the ratio W_1/W_1 we raise to the tenth power:

$$\frac{W_1}{W_2} = 10^{0.5} = 3.162.$$

The first earthquake was just over three times as large as the second one.

60. We know that the y-intercept of the line is at (0,1), so we need one other point to determine the equation of the line. We observe that it intersects the graph of $f(x) = 10^x$ at the point $x = \log 2$. The y-coordinate of this point is then

$$y = 10^x = 10^{\log 2} = 2,$$

so $(\log 2, 2)$ is the point of intersection. We can now find the slope of the line:

$$m = \frac{2-1}{\log 2 - 0} = \frac{1}{\log 2}.$$

Plugging this into the point-slope formula for a line, we have

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{\log 2}(x - 0) \\ y &= \frac{1}{\log 2}x + 1 \approx 3.3219x + 1. \end{aligned}$$

61. The function e^x has a vertical intercept of 1, so must be A. The function $\ln x$ has an x-intercept of 1, so must be D. The graphs of x^2 and $x^{1/2}$ go through the origin. The graph of $x^{1/2}$ is concave down so it corresponds to graph C and the graph of x^2 is concave up so it corresponds to graph B.

62. Yes, $\ln(\ln(x))$ means take the ln of the value of the function $\ln x$. On the other hand, $\ln^2(x)$ means take the function $\ln x$ and square it. For example, consider each of these functions evaluated at e. Since $\ln e = 1$, $\ln^2 e = 1^2 = 1$, but $\ln(\ln(e)) = \ln(1) = 0$. See the graphs in Figure 1.40. (Note that $\ln(\ln(x))$ is only defined for $x > 1$.)

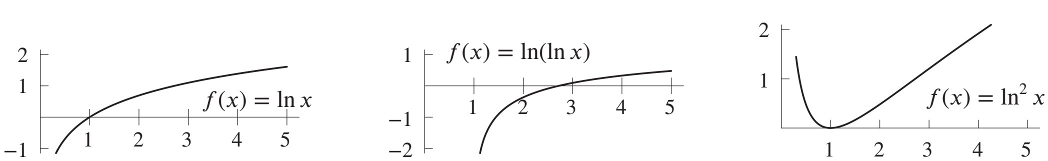


Figure 1.40

63. (a) The y-intercept of $h(x) = \ln(x + a)$ is $h(0) = \ln a$. Thus increasing a increases the y-intercept.
(b) The x-intercept of $h(x) = \ln(x + a)$ is where $h(x) = 0$. Since this occurs where $x + a = 1$, or $x = 1 - a$, increasing a moves the x-intercept to the left.

64. The vertical asymptote is where $x + a = 0$, or $x = -a$. Thus increasing a moves the vertical asymptote to the left.

65. (a) The y-intercept of $g(x) = \ln(ax + 2)$ is $g(0) = \ln 2$. Thus increasing a does not effect the y-intercept.
(b) The x-intercept of $g(x) = \ln(ax + 2)$ is where $g(x) = 0$. Since this occurs where $ax + 2 = 1$, or $x = -1/a$, increasing a moves the x-intercept toward the origin. (The intercept is to the left of the origin if $a > 0$ and to the right if $a < 0$.)

66. The vertical asymptote is where $x + 2 = 0$, or $x = -2$, so increasing a does not effect the vertical asymptote.

67. The vertical asymptote is where $ax + 2 = 0$, or $x = -2/a$. Thus increasing a moves the vertical asymptote toward the origin. (The asymptote is to the left of the origin for $a > 0$ and to the right of the origin for $a < 0$.)

68. Properties of logs give

$$\ln(P_0e^{kt}) = \ln P_0 + \ln e^{kt} = \ln P_0 + kt.$$

Therefore

$$\begin{aligned}\frac{\ln f(t_2) - \ln f(t_1)}{t_2 - t_1} &= \frac{\ln(P_0e^{kt_2}) - \ln(P_0e^{kt_1})}{t_2 - t_1}\\ &= \frac{\ln P_0 + kt_2 - \ln P_0 - kt_1}{t_2 - t_1}\\ &= \frac{k(t_2 - t_1)}{t_2 - t_1} = k.\end{aligned}$$

Strengthen Your Understanding

69. The function $-\log|x|$ is even, since $|-x| = |x|$, which means $-\log|-x| = -\log|x|$.

70. We have

$$\ln(100x) = \ln(100) + \ln x.$$

In general, $\ln(100x) \neq 100 \cdot \ln x$.

71. As seen from the graphs of the functions, this inequality holds only for $x > 1$. The functions cross at the point $(1, 0)$ and the inequality is reversed for $x < 1$.

72. The correct property says that the logarithm of a product is the sum of the logarithms, that is

$$\ln(AB) = \ln A + \ln B.$$

We may also check that the given property is not correct by taking $A = B = 1$. Then the statement gives us the incorrect statement:

$$\ln 2 = 2\ln 1 = 0.$$

73. One of the properties of logarithms states that the logarithm of a product is the sum of the logarithms, that is

$$\ln(AB) = \ln A + \ln B.$$

There is no property of logarithms that states the logarithm of a sum is the sum of logarithms. We can check by taking $A = B = 1$. Then the statement gives us the incorrect statement:

$$\ln 2 = \ln 1 + \ln 1 = 0.$$

74. The functions $y = \ln x$ and $y = e^x$ are inverses of each other, but not reciprocals of each other. As their graphs show, $y = 1/\ln x$ and $y = e^x$ don't even have the same domain.

75. The function $y = 0.7\log x$ works, as it scales down each value of $y = \log x$ for $x > 1$. The graph of $y = 0.7\log x$ is below the graph of $y = \log x$ as long as $x > 1$.

76. One possibility is $f(x) = -x$, because $\ln(-x)$ is only defined if $-x > 0$.

77. One possibility is $f(x) = \ln(x - 3)$.

78. True, as seen from the graph.

79. False, since $\log(x - 1) = 0$ if $x - 1 = 1$, so $x = 2$.

80. False. The inverse function is $y = 10^x$.

81. False, since $ax + b = 0$ if $x = -b/a$. Thus $y = \ln(ax + b)$ has a vertical asymptote at $x = -b/a$.

Solutions for Section 1.5

Exercises

1. See Figure 1.41.

$$\begin{aligned} \sin\left(\frac{3\pi}{2}\right) &= -1 \quad \text{is negative.} \\ \cos\left(\frac{3\pi}{2}\right) &= 0 \\ \tan\left(\frac{3\pi}{2}\right) & \quad \text{is undefined.} \end{aligned}$$

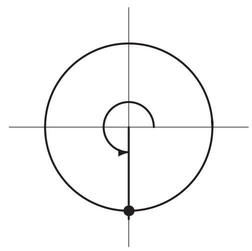


Figure 1.41

2. See Figure 1.42.

$$\begin{aligned} \sin(2\pi) &= 0 \\ \cos(2\pi) &= 1 \text{ is positive.} \\ \tan(2\pi) &= 0 \end{aligned}$$

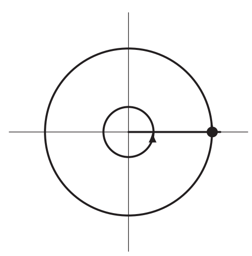


Figure 1.42

3. See Figure 1.43.

$$\begin{aligned} \sin\frac{\pi}{4} &\quad \text{is positive} \\ \cos\frac{\pi}{4} &\quad \text{is positive} \\ \tan\frac{\pi}{4} &\quad \text{is positive} \end{aligned}$$

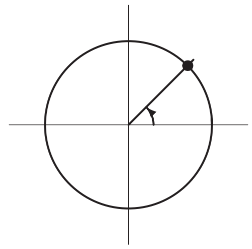


Figure 1.43

4. See Figure 1.44.

$$\begin{aligned}\sin 3\pi &= 0\\ \cos 3\pi &= -1 \text{ is negative}\\ \tan 3\pi &= 0\end{aligned}$$

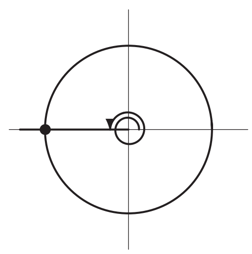


Figure 1.44

5. See Figure 1.45.

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &\text{ is positive.}\\ \cos\left(\frac{\pi}{6}\right) &\text{ is positive.}\\ \tan\left(\frac{\pi}{6}\right) &\text{ is positive.}\end{aligned}$$

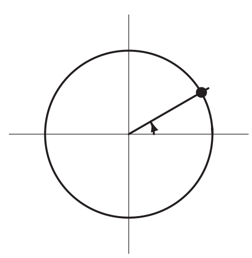


Figure 1.45

6. See Figure 1.46.

$$\begin{aligned}\sin\frac{4\pi}{3} &\quad \text{is negative}\\ \cos\frac{4\pi}{3} &\quad \text{is negative}\\ \tan\frac{4\pi}{3} &\quad \text{is positive}\end{aligned}$$

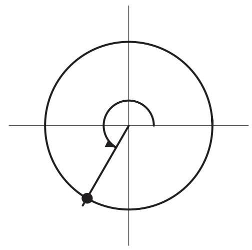


Figure 1.46

7. See Figure 1.47.

$$\sin\left(\frac{-4\pi}{3}\right) \text{ is positive.}$$
$$\cos\left(\frac{-4\pi}{3}\right) \text{ is negative.}$$
$$\tan\left(\frac{-4\pi}{3}\right) \text{ is negative.}$$

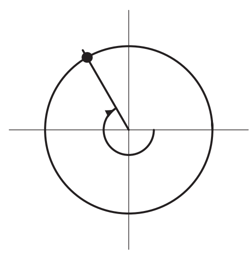


Figure 1.47

8. $4 \text{ radians} \cdot \dfrac{180^\circ}{\pi \text{ radians}} = \left(\dfrac{720}{\pi}\right)^\circ \approx 240^\circ$. See Figure 1.48.

$$\begin{aligned} \sin 4 &\quad \text{is negative} \\ \cos 4 &\quad \text{is negative} \\ \tan 4 &\quad \text{is positive.} \end{aligned}$$

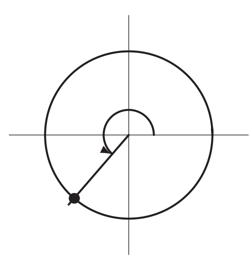


Figure 1.48

9. $-1 \text{ radian} \cdot \frac{180°}{\pi \text{ radians}} = -\left(\frac{180°}{\pi}\right) \approx -60°$. See Figure 1.49.

$$\begin{array}{ll} \sin(-1) & \text{is negative} \\ \cos(-1) & \text{is positive} \\ \tan(-1) & \text{is negative.} \end{array}$$

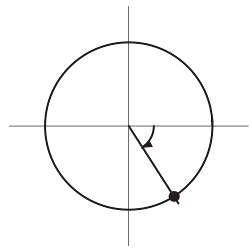


Figure 1.49

10. The period is $2\pi/3$, because when t varies from 0 to $2\pi/3$, the quantity $3t$ varies from 0 to 2π. The amplitude is 7, since the value of the function oscillates between -7 and 7.

11. The period is $2\pi/(1/4) = 8\pi$, because when u varies from 0 to 8π, the quantity $u/4$ varies from 0 to 2π. The amplitude is 3, since the function oscillates between 2 and 8.

12. The period is $2\pi/2 = \pi$, because as x varies from $-\pi/2$ to $\pi/2$, the quantity $2x + \pi$ varies from 0 to 2π. The amplitude is 4, since the function oscillates between 4 and 12.

13. The period is $2\pi/\pi = 2$, since when t increases from 0 to 2, the value of πt increases from 0 to 2π. The amplitude is 0.1, since the function oscillates between 1.9 and 2.1.

14. This graph is a sine curve with period 8π and amplitude 2, so it is given by $f(x) = 2\sin\left(\frac{x}{4}\right)$.

15. This graph is a cosine curve with period 6π and amplitude 5, so it is given by $f(x) = 5\cos\left(\frac{x}{3}\right)$.

16. This graph is an inverted sine curve with amplitude 4 and period π, so it is given by $f(x) = -4\sin(2x)$.

17. This graph is an inverted cosine curve with amplitude 8 and period 20π, so it is given by $f(x) = -8\cos\left(\frac{x}{10}\right)$.

18. This graph has period 6, amplitude 5 and no vertical or horizontal shift, so it is given by

$$f(x) = 5\sin\left(\frac{2\pi}{6}x\right) = 5\sin\left(\frac{\pi}{3}x\right).$$

19. The graph is a cosine curve with period $2\pi/5$ and amplitude 2, so it is given by $f(x) = 2\cos(5x)$.

20. The graph is an inverted sine curve with amplitude 1 and period 2π, shifted up by 2, so it is given by $f(x) = 2 - \sin x$.

21. This can be represented by a sine function of amplitude 3 and period 18. Thus,

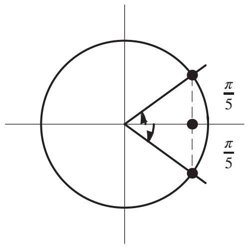
$$f(x) = 3\sin\left(\frac{\pi}{9}x\right).$$

22. This graph is the same as in Problem 14 but shifted up by 2, so it is given by $f(x) = 2\sin\left(\frac{x}{4}\right) + 2$.

23. This graph has period 8, amplitude 3, and a vertical shift of 3 with no horizontal shift. It is given by

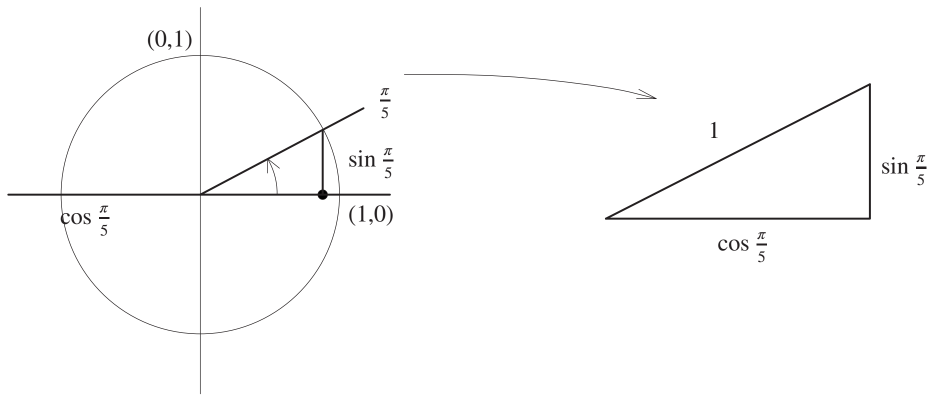
$$f(x) = 3 + 3\sin\left(\frac{2\pi}{8}x\right) = 3 + 3\sin\left(\frac{\pi}{4}x\right).$$

24.



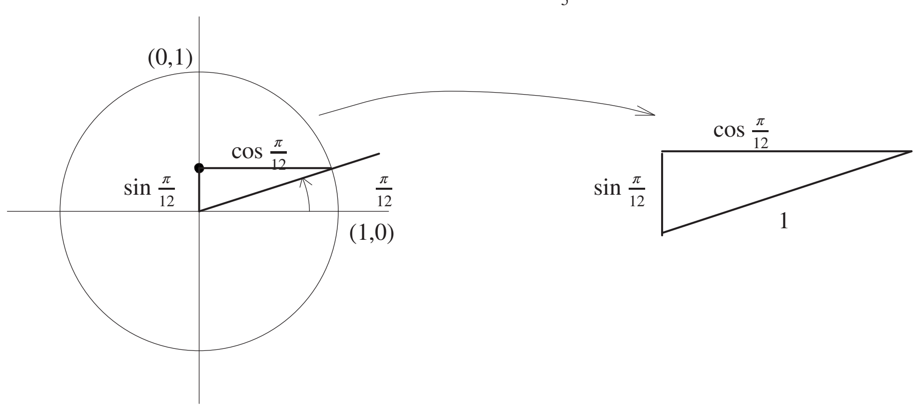
$$\cos\left(-\frac{\pi}{5}\right) = \cos\frac{\pi}{5} \quad \text{(by picture)}$$
$$= 0.809.$$

25.



By the Pythagorean Theorem, $(\cos\frac{\pi}{5})^2 + (\sin\frac{\pi}{5})^2 = 1^2$;
so $(\sin\frac{\pi}{5})^2 = 1 - (\cos\frac{\pi}{5})^2$, and $\sin\frac{\pi}{5} = \sqrt{1-(\cos\frac{\pi}{5})^2} = \sqrt{1-(0.809)^2} \approx 0.588$.
We take the positive square root since by the picture we know that $\sin\frac{\pi}{5}$ is positive.

26.



By the Pythagorean Theorem, $(\cos\frac{\pi}{12})^2 + (\sin\frac{\pi}{12})^2 = 1^2$; so $(\cos\frac{\pi}{12})^2 = 1 - (\sin\frac{\pi}{12})^2$ and $\cos\frac{\pi}{12} = \sqrt{1-(\sin\frac{\pi}{12})^2} = \sqrt{1-(0.259)^2} \approx 0.966$. We take the positive square root since by the picture we know that $\cos\frac{\pi}{12}$ is positive.

27. Since $\sin\theta$ is an odd function, $\sin\left(-\frac{\pi}{12}\right) = -\sin\left(\frac{\pi}{12}\right) = -0.259$.

28. Since $\tan\theta = \sin\theta/\cos\theta$, we need to find $\cos\frac{\pi}{12}$ first. We can use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to find

$$\cos\frac{\pi}{12} = \pm\sqrt{1-0.259^2} = \pm 0.966.$$

Since the angle $\theta = \frac{\pi}{12}$ lies in the first quadrant, $\cos\theta = 0.966$. Thus,

$$\tan\frac{\pi}{12} = \frac{0.259}{0.966} = 0.268.$$

29. Since $\tan\theta = \sin\theta/\cos\theta$, we need to find $\sin\frac{\pi}{5}$ first. We can use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to find

$$\sin\frac{\pi}{5} = \pm\sqrt{1 - 0.809^2} = \pm 0.588.$$

Since the angle $\theta = \dfrac{\pi}{5}$ lies in the first quadrant, $\sin\frac{\pi}{5} = 0.588$. Thus,

$$\tan\frac{\pi}{5} = \frac{0.588}{0.809} = 0.727.$$

30. We first divide by 5 and then use inverse sine:

$$\begin{aligned} \frac{2}{5} &= \sin(3x) \\ \sin^{-1}(2/5) &= 3x \\ x &= \frac{\sin^{-1}(2/5)}{3} \approx 0.1372. \end{aligned}$$

There are infinitely many other possible solutions since the sine is periodic.

31. We first isolate $\cos(2x+1)$ and then use inverse cosine:

$$\begin{aligned} 1 &= 8\cos(2x+1) - 3 \\ 4 &= 8\cos(2x+1) \\ 0.5 &= \cos(2x+1) \\ \cos^{-1}(0.5) &= 2x+1 \\ x &= \frac{\cos^{-1}(0.5) - 1}{2} \approx 0.0236. \end{aligned}$$

There are infinitely many other possible solutions since the cosine is periodic.

32. We first isolate $\tan(5x)$ and then use inverse tangent:

$$\begin{aligned} 8 &= 4\tan(5x) \\ 2 &= \tan(5x) \\ \tan^{-1} 2 &= 5x \\ x &= \frac{\tan^{-1} 2}{5} = 0.221. \end{aligned}$$

There are infinitely many other possible solutions since the tangent is periodic.

33. We first isolate $(2x+1)$ and then use inverse tangent:

$$\begin{aligned} 1 &= 8\tan(2x+1) - 3 \\ 4 &= 8\tan(2x+1) \\ 0.5 &= \tan(2x+1) \\ \arctan(0.5) &= 2x+1 \\ x &= \frac{\arctan(0.5) - 1}{2} = -0.268. \end{aligned}$$

There are infinitely many other possible solutions since the tangent is periodic.

34. We first isolate $\sin(5x)$ and then use inverse sine:

$$\begin{aligned} 8 &= 4\sin(5x) \\ 2 &= \sin(5x). \end{aligned}$$

But this equation has no solution since $-1 \le \sin(5x) \le 1$.

35. The earth makes one revolution around the sun in one year, so its period is one year.

36. The moon makes one revolution around the earth in about 27.3 days, so its period is 27.3 days $\approx$ one month.

37. 200 revolutions per minute is $\frac{1}{200}$ minutes per revolution, so the period is $\frac{1}{200}$ minutes, or 0.3 seconds.

38. Using the fact that 1 revolution $= 2\pi$ radians and 1 minute $= 60$ seconds, we have

$$\begin{aligned} 200\frac{\text{rev}}{\text{min}} &= (200)\cdot 2\pi\frac{\text{rad}}{\text{min}} = 200\cdot 2\pi\frac{1}{60}\frac{\text{rad}}{\text{sec}} \\ &\approx \frac{(200)(6.283)}{60} \\ &\approx 20.94 \text{ radians per second.} \end{aligned}$$

Similarly, 500 rpm is equivalent to 52.36 radians per second.

39. The ramp in Figure 1.50 rises 1 ft over a horizontal distance of x ft.

(a) For a 1 ft rise over 12 ft, the angle in radians is $\theta = \arctan(1/12) = 0.0831$. To find the angle in degrees, multiply by $180/\pi$. Hence

$$\theta = \frac{180}{\pi}\arctan\frac{1}{12} = 4.76°.$$

(b) We have

$$\theta = \frac{180}{\pi}\arctan\frac{1}{8} = 7.13°.$$

(c) We have

$$\theta = \frac{180}{\pi}\arctan\frac{1}{20} = 2.86°.$$

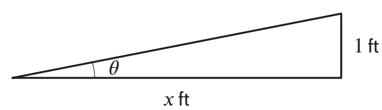


Figure 1.50

Problems

40. (a) The period is 2π.

(b) After π, the values of $\cos 2\theta$ repeat, but the values of $2\sin\theta$ do not (in fact, they repeat but flipped over the x-axis). After another π, that is after a total of 2π, the values of $\cos 2\theta$ repeat *again*, and now the values of $2\sin\theta$ repeat also, so the function $2\sin\theta + 3\cos 2\theta$ repeats at that point.

41. All three functions have amplitude 2 and period 2π. Each is just a horizontal shift of the others.

(a) We see that $h(t)$ looks like a cosine graph with amplitude 2 and period 2π, but shifted right by $\pi/2$. Thus, $h(t) = 2\cos(t - \pi/2)$.

(b) We see that $f(t)$ has and amplitude of 2 and a period of 2π. Furthermore, since $f(0) = 2$, we know that $f(t) = 2\cos t$.

(c) We see that $g(t)$ looks like a cosine graph with amplitude 2 and period 2π, but shifted left by $\pi/2$. Thus, $g(t) = 2\cos(t + \pi/2)$.

42. (a) The parameter A is the amplitude of the function. Since the functions oscillate between -20 and $+20$, we have $A = 20$.

(b) We have $B = 2\pi/P$, where P is the period of the function. From the graphs, we see that the function repeats every 100 units, so the period is 100. Thus $B = 2\pi/100 = 0.0628$.

(c) Since $\cos t$ has a maximum at $t = 0$, the function $A\cos((2\pi/100)t)$ also achieves its maximum at $t = 0$. Thus $h = 0$ corresponds to graph (II).

The three graphs (I), (III), (IV) are all obtained by shifting graph (II) horizontally to the right. The size of the shift, h, can be determined by checking how far the first maximum on the graph has been shifted from the vertical axis. Graph (I) is shifted slightly more than 50, so graph (I) corresponds to $h = 60$. Similarly, graph (III) corresponds to $h = 20$ and graph (IV) to $h = 50$.

43. The graph is a stretched sine function, so it starts at the midline, $y = 0$, when $x = 0$.

The amplitude is k, so the sinusoidal graph oscillates between a minimum of $-k$ and a maximum of k.

Since k is positive, the function begins by increasing. The period is 2π. See Figure 1.51.

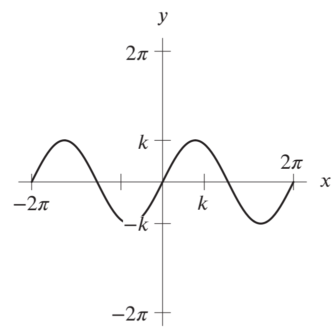


Figure 1.51

44. The graph is a stretched cosine function reflected about the x-axis, so it starts at its minimum when $x = 0$.
The amplitude is k, so the sinusoidal graph oscillates between a minimum of $-k$ and a maximum of k.
The period is 2π. See Figure 1.52.

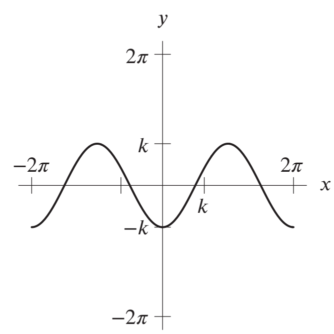


Figure 1.52

45. The graph is a stretched cosine function shifted vertically, so it starts at its maximum when $x = 0$.

We begin by finding the midline, which is k, since the sine function is shifted up by k. The amplitude is k, so the sinusoidal graph oscillates between a minimum of $k - k = 0$ and a maximum of $k + k = 2k$.

The period is 2π. See Figure 1.53.

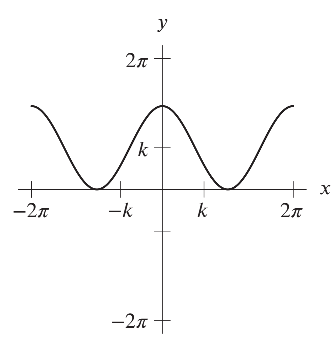


Figure 1.53

46. The graph is a stretched sine function shifted vertically, so it starts at its midline when $x = 0$.

We begin by finding the midline, which is $-k$, since the sine function is shifted up by $-k$ (down by k). The amplitude is k, so the sinusoidal graph oscillates between a minimum of $-k - k = -2k$ and a maximum of $-k + k = 0$.

Since k is positive, the function begins by increasing. The period is 2π. See Figure 1.54.

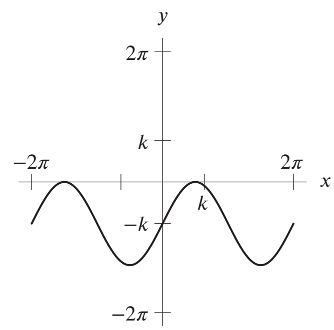


Figure 1.54

47. The graph is a stretched sine function shifted horizontally. Since we have $\sin(x-k)$, the function is shifted k units to the right, so the function takes on the value of its midline, 0, when $x = k$.

The amplitude is k, so the sinusoidal graph oscillates between a minimum of $-k$ and a maximum of k.

Since k is positive, the function is increasing at $x = k$. The period is 2π. See Figure 1.55.

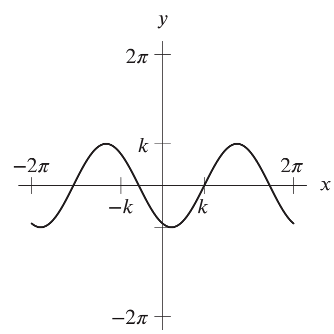


Figure 1.55

48. The graph is a stretched cosine function shifted horizontally. Since we have $\cos(x+k)$, the function is shifted k units to the left, so the function takes on its maximum value when $x = -k$.

The amplitude is k, so the sinusoidal graph oscillates between a minimum of $-k$ and a maximum of k.

The period is 2π. See Figure 1.56.

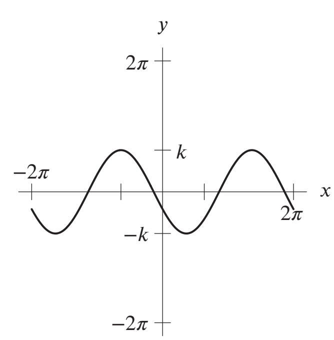


Figure 1.56

49. The graph is a vertically stretched sine function horizontally compressed by a factor of $2\pi/k$. The function takes on the value of its midline, 0, when $x = k$.

The amplitude is k, so the sinusoidal graph oscillates between a minimum of $-k$ and a maximum of k.

Since k is positive, the function is increasing at $x = 0$. The period is $2\pi/(2\pi/k) = k$. See Figure 1.57.

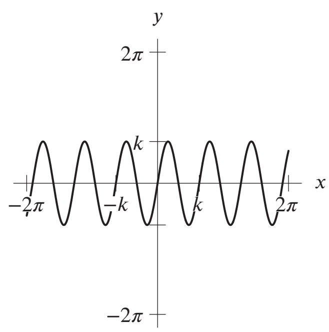


Figure 1.57

50. The parameter C gives the midline of the graph. The minimum is at about 6 and the maximum is about 30; the midline is the average value. Thus

$$C = \frac{6+30}{2} = 18°\text{C}.$$

51. The parameter A gives the amplitude of the oscillation. The midline is at about 18 and the maximum is about 30, so the amplitude is the difference

$$A = 30 - 18 = 12°\text{C}.$$

52. We have $B = 2\pi/P$ where P is the period of the function. Since the period is 12 months, we have $B = 2\pi/12 = \pi/6$.

53. Since the maximum value of the function appears at about $t = 7.5$, we see that the graph is obtained by horizontally shifting the graph of $A\cos(Bt) + C$ to the right by 7.5 units. Thus $h = 7.5$ months.

54. The parameter C gives the midline of the graph. The minimum is at about 175 and the maximum is about 725; the midline is the average value. Thus

$$C = \frac{175+725}{2} = 450 \text{ thousand ft}^3/\text{sec}.$$

55. The parameter A gives the amplitude of the oscillation. The midline is about 450 and the maximum is about 725; the amplitude is the difference. Thus

$$A = 725 - 450 = 275 \text{ thousand ft}^3/\text{sec}.$$

56. We have $B = 2\pi/P$ where P is the period of the function. Since the period is 12 months, we have $B = 2\pi/12 = \pi/6$.

57. Since the graph increases across the midline at about $t = 1$, we see that it is obtained by horizontally shifting the graph $A\sin(Bt) + C$ to the right by 1 unit. Thus $h = 1$ month.

58. Over the one-year period, the average value is about 75° and the amplitude of the variation is about $\frac{90-60}{2} = 15°$. The function assumes its minimum value right at the beginning of the year, so we want a negative cosine function. Thus, for t in years, we have the function

$$f(t) = 75 - 15\cos\left(\frac{2\pi}{12}t\right).$$

(Many other answers are possible, depending on how you read the chart.)

59. $\sin x^2$ is by convention $\sin(x^2)$, which means you square the x first and then take the sine.

$\sin^2 x = (\sin x)^2$ means find $\sin x$ and then square it.

$\sin(\sin x)$ means find $\sin x$ and then take the sine of that.

Expressing each as a composition: If $f(x) = \sin x$ and $g(x) = x^2$, then

$\sin x^2 = f(g(x))$
$\sin^2 x = g(f(x))$
$\sin(\sin x) = f(f(x))$.

60. Suppose P is at the point $(3\pi/2, -1)$ and Q is at the point $(5\pi/2, 1)$. Then

$$\text{Slope} = \frac{1-(-1)}{5\pi/2 - 3\pi/2} = \frac{2}{\pi}.$$

If P had been picked to the right of Q, the slope would have been $-2/\pi$.

61. (a) See Figure 1.58.

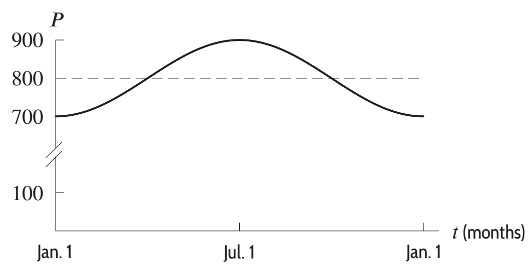


Figure 1.58

(b) Average value of population $= \frac{700+900}{2} = 800$, amplitude $= \frac{900-700}{2} = 100$, and period $= 12$ months, so $B = 2\pi/12 = \pi/6$. Since the population is at its minimum when $t = 0$, we use a negative cosine:

$$P = 800 - 100\cos\left(\frac{\pi t}{6}\right).$$

62. We use a cosine of the form

$$H = A\cos(Bt) + C$$

and choose B so that the period is 24 hours, so $2\pi/B = 24$ giving $B = \pi/12$.

The temperature oscillates around an average value of 60° F, so $C = 60$. The amplitude of the oscillation is 20° F. To arrange that the temperature be at its lowest when $t = 0$, we take A negative, so $A = -20$. Thus

$$A = 60 - 20\cos\left(\frac{\pi}{12}t\right).$$

63. Depth $= 7 + 1.5\sin\left(\frac{\pi}{3}t\right)$

64. (a) Beginning at time $t = 0$, the voltage will have oscillated through a complete cycle when $\cos(120\pi t) = \cos(2\pi)$, hence when $t = \frac{1}{60}$ second. The period is $\frac{1}{60}$ second.
(b) V_0 represents the amplitude of the oscillation.
(c) See Figure 1.59.

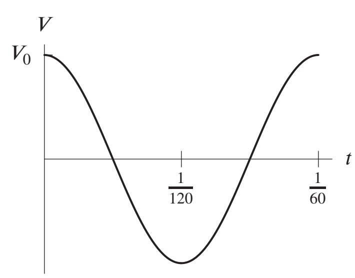


Figure 1.59

65. (a) When the time is t hours after 6 am, the solar panel outputs $f(t) = P(\theta(t))$ watts. So,

$$f(t) = 10\sin\left(\frac{\pi}{14}t\right)$$

where $0 \le t \le 14$ is the number of hours after 6 am.

(b) The graph of $f(t)$ is in Figure 1.60:

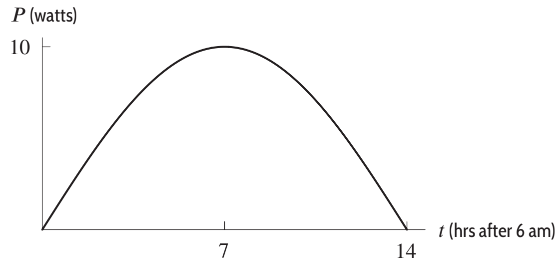


Figure 1.60

(c) The power output is greatest when $\sin(\pi t/14) = 1$. Since $0 \le \pi t/14 \le \pi$, the only point in the domain of f at which $\sin(\pi t/14) = 1$ is when $\pi t/14 = \pi/2$. Therefore, the power output is greatest when $t = 7$, that is, at 1 pm. The output at this time will be $f(7) = 10$ watts.

(d) On a typical winter day, there are 9 hours of sun instead of the 14 hours of sun. So, if t is the number of hours since 8 am, the angle between a solar panel and the sun is

$$\phi = \frac{14}{9}\theta = \frac{\pi}{9}t \quad \text{where } 0 \le t \le 9.$$

The solar panel outputs $g(t) = P(\phi(t))$ watts:

$$g(t) = 10\sin\left(\frac{\pi}{9}t\right)$$

where $0 \le t \le 9$ is the number of hours after 8 am.

66. The function R has period of π, so its graph is as shown in Figure 1.61. The maximum value of the range is v_0^2/g and occurs when $\theta = \pi/4$.

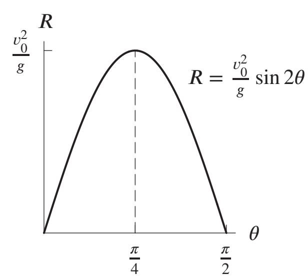


Figure 1.61

67. (a) $f(t) = -0.5 + \sin t, \quad g(t) = 1.5 + \sin t, \quad h(t) = -1.5 + \sin t, \quad k(t) = 0.5 + \sin t.$

(b) The values of $g(t)$ are one more than the values of $k(t)$, so $g(t) = 1 + k(t)$. This happens because $g(t) = 1.5 + \sin t = 1 + 0.5 + \sin t = 1 + k(t)$.

(c) Since $-1 \le \sin t \le 1$, adding 1.5 everywhere we get $0.5 \le 1.5 + \sin t \le 2.5$ and since $1.5 + \sin t = g(t)$, we get $0.5 \le g(t) \le 2.5$. Similarly, $-2.5 \le -1.5 + \sin t = h(t) \le -0.5$.

68. (a) The period of the tides is $2\pi/0.5 = 4\pi = 12.566$ hours.

(b) The boat is afloat provided the water is deeper than 2.5 meters, so we need

$$d(t) = 5 + 4.6\sin(0.5t) > 2.5.$$

Figure 1.62 is a graph of $d(t)$, with time t in hours since midnight, $0 \le t \le 24$. The boat leaves at $t = 12$ (midday). To find the latest time the boat can return, we need to solve the equation $d(t) = 5 + 4.6\sin(0.5t) = 2.5$.

A quick way to estimate the solution is to trace along the line $y = 2.5$ in Figure 1.62 until we get to the first point of intersection to the right of $t = 12$. The value we want is about $t = 20$. Thus the water remains deep enough until about 8 pm.

To find t analytically, we solve

$$\begin{aligned}
5 + 4.6\sin(0.5t) &= 2.5 \\
\sin(0.5t) &= -\frac{2.5}{4.6} = -0.5435 \\
t &= \frac{1}{0.5}\arcsin(-0.5435) = -1.149.
\end{aligned}$$

This is the value of t immediately to the left of the vertical axis. The water is also 2.5 meters deep one period later at $t = -1.149 + 12.566 = 11.417$. This is shortly before the boat leaves, while the water is rising. We want the next time the time the water is this depth.

The water was at its deepest (that is, $d(t)$ was a maximum) when $t = 12.566/4 = 3.142$. From the figure, the time between when the water was 2.5 meters and when it was deepest was $3.142 + 1.149 = 4.291$ hours. Thus, the value of t that we want is

$$t = 11.417 + 2 \cdot 4.291 = 19.999.$$

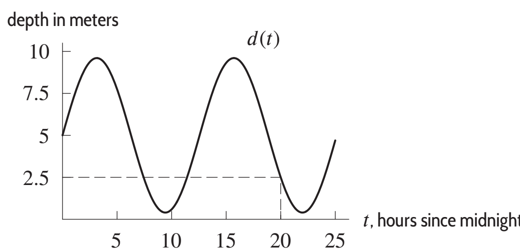


Figure 1.62

69. (a) D = the average depth of the water.
(b) A = the amplitude $= 15/2 = 7.5$.
(c) Period = 12.4 hours. Thus $(B)(12.4) = 2\pi$ so $B = 2\pi/12.4 \approx 0.507$.
(d) C is the time of a high tide.

70. Since b is a positive constant, f is a vertical shift of $\sin t$ where the midline lies above the t-axis. So f matches Graph C.

Function g is the sum of $\sin t$ plus a linear function $at + b$. We suspect then that the graph of g might periodically oscillate about a line $at + b$, just like the graph of $\sin t$ oscillates about its midline. When adding $at + b$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, at + b)$ that lies both on the graph of g, and on the line $at + b$. See Figure 1.63. So g matches Graph B.

Function h is the sum of $\sin t$ plus an increasing exponential function $e^{ct} + d$. We suspect then that the graph of g might periodically oscillate about the graph of $e^{ct} + d$, just like the graph of $\sin t$ oscillates about its midline. When adding $e^{ct} + d$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, e^{ct} + d)$ that lies both on the graph of h, and on the graph of $e^{ct} + d$. See Figure 1.63. So h matches Graph A. Note that the oscillations on the graph of h may not be visible for all t values.

Function r is the sum of $\sin t$ plus a decreasing exponential function $-e^{ct} + b$. We suspect then that the graph of r might periodically oscillate about the graph of $-e^{ct} + b$, just like the graph of $\sin t$ oscillates about its midline. When adding $-e^{ct} + b$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, -e^{ct} + b)$ that lies both on the graph of r, and on the graph of $-e^{ct} + b$. See Figure 1.63. So r matches Graph D. Note that the oscillations on the graph of r may not be visible for all t values.

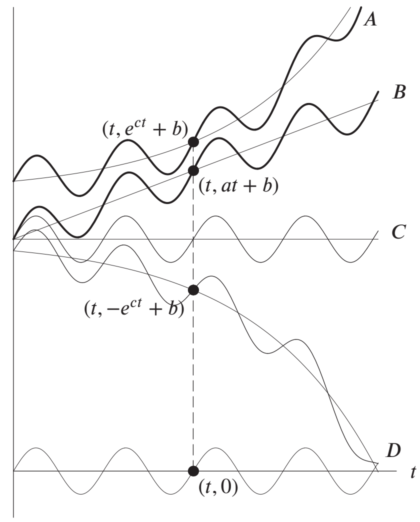


Figure 1.63

71. (a) The monthly mean CO_2 increased about 10 ppm between December 2005 and December 2010. This is because the black curve shows that the December 2005 monthly mean was about 381 ppm, while the December 2010 monthly mean was about 391 ppm. The difference between these two values, $391 - 381 = 10$, gives the overall increase.

(b) The average rate of increase is given by

$$\text{Average monthly increase of monthly mean} = \frac{391 - 381}{60 - 0} = \frac{1}{6} \text{ ppm/month.}$$

This tells us that the slope of a linear equation approximating the black curve is $1/6$. Since the vertical intercept is about 381, a possible equation for the approximately linear black curve is

$$y = \frac{1}{6}t + 381,$$

where t is measured in months since December 2005.

(c) The period of the seasonal CO_2 variation is about 12 months since this is approximately the time it takes for the function given by the blue curve to complete a full cycle. The amplitude is about 3.5 since, looking at the blue curve, the average distance between consecutive maximum and minimum values is about 7 ppm. So a possible sinusoidal function for the seasonal CO_2 cycle is

$$y = 3.5 \sin\left(\frac{\pi}{6}t\right).$$

(d) Taking $f(t) = 3.5 \sin\left(\frac{\pi}{6}t\right)$ and $g(t) = \frac{1}{6}t + 381$, we have

$$h(t) = 3.5 \sin\left(\frac{\pi}{6}t\right) + \frac{1}{6}t + 381.$$

See Figure 1.64.

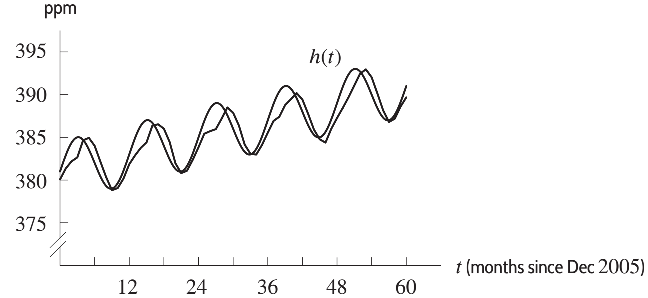


Figure 1.64

72. Figure 1.65 shows that the cross-sectional area is one rectangle of area hw and two triangles. Each triangle has height h and base x, where

$$\frac{h}{x} = \tan\theta \quad \text{so} \quad x = \frac{h}{\tan\theta}.$$

$$\text{Area of triangle} = \frac{1}{2}xh = \frac{h^2}{2\tan\theta}$$

$$\begin{aligned} \text{Total area} &= \text{Area of rectangle} + 2(\text{Area of triangle}) \\ &= hw + 2\cdot\frac{h^2}{2\tan\theta} = hw + \frac{h^2}{\tan\theta}. \end{aligned}$$

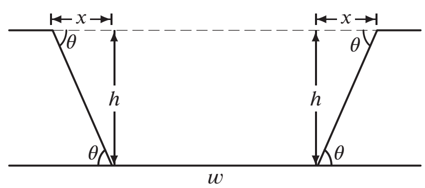


Figure 1.65

73. (a) Two solutions: 0.4 and 2.7. See Figure 1.66.
(b) arcsin(0.4) is the first solution approximated above; the second is an approximation to $\pi - \arcsin(0.4)$.
(c) By symmetry, there are two solutions: -0.4 and -2.7.
(d) $-0.4 \approx -\arcsin(0.4)$ and $-2.7 \approx -(\pi - \arcsin(0.4)) = \arcsin(0.4) - \pi$.

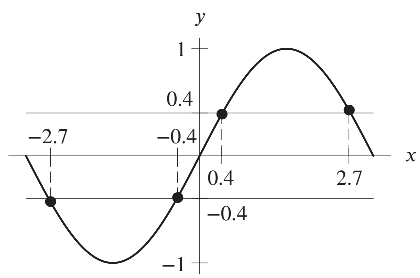


Figure 1.66

74. (a) A table of values for $g(x)$ is:

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
$\arccos x$	3.14	2.50	2.21	1.98	1.77	1.57	1.37	1.16	0.93	0.64	0

(b) See Figure 1.67.

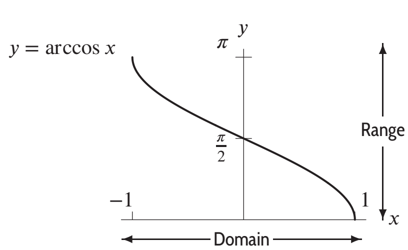


Figure 1.67

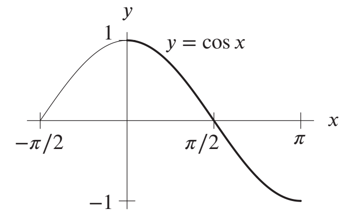


Figure 1.68

(c) The domain of arccos is $-1 \leq x \leq 1$, because its inverse, cosine, takes all values from -1 to 1. The domains of arccos and arcsin are the same because their inverses, sine and cosine, have the same range.
(d) Figure 1.67 shows that the range of $y = \arccos x$ is $0 \leq \theta \leq \pi$.
(e) The range of an inverse function is the domain of the original function. The arcsine is the inverse function to the piece of the sine having domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Hence, the range of the arcsine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. But the piece of the cosine having domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ does not have an inverse, because there are horizontal lines that intersect its graph twice. Instead, we define arccosine to be the inverse of the piece of cosine having domain $[0, \pi]$, so the range of arccosine is $[0, \pi]$, which is different from the range of arcsine. See Figure 1.68.

Strengthen Your Understanding

75. Scaling the output (that is, the function values, as in $y = 3\cos x$) does not change the period but changes the amplitude. Scaling the input (that is, the independent variable, as in $y = \cos 3x$) changes the period of the function but not the amplitude. So f has the same period, 2π, as $\cos x$, but g has a period smaller than 2π. See Figure 1.69.

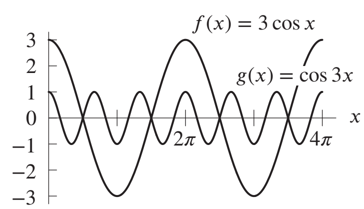


Figure 1.69

76. Increasing the value of B decreases the period. For example, $f(x) = \sin x$ has period 2π, whereas $g(x) = \sin(2x)$ has period π.

77. The functions $f(x) = \sin x$ and $g(x) = \cos x$ both have period 2π. The product, fg, of these two functions is periodic, but has a smaller period, π. See Figure 1.70.

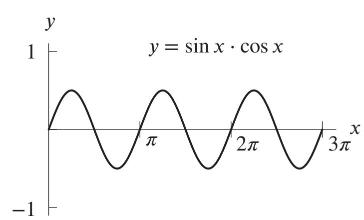


Figure 1.70

78. The maximum value of $A\sin(Bx)$ is A, so the maximum value of $y = A\sin(Bx) + C$ is $y = A + C$.

79. For $B > 0$, the period of $y = \sin(Bx)$ is $2\pi/B$. Thus, we want

$$\frac{2\pi}{B} = 23 \quad \text{so} \quad B = \frac{2\pi}{23}.$$

The function is $f(x) = \sin(2\pi x/23)$

80. The midline is $y = (1200 + 2000)/2 = 1600$ and the amplitude is $y = (2000 - 1200)/2 = 400$, so a possible function is

$$f(x) = 400(\cos x) + 1600.$$

81. True. All functions in this family have period 2π since the independent variable, x, (the input) is not scaled by a constant. The constant a varies the amplitude of the functions in the family, but not the period.

82. False. Scaling the independent variable by a constant varies the period of the functions in the family. For a given value of a, the period of the corresponding function in the family is $2\pi/a$.

83. False. The function $\cos\theta$ is decreasing and the function $\sin\theta$ is increasing. Thus, the function $-\sin\theta$ is decreasing over this interval, which means f is decreasing.

84. False. The period is $2\pi/(0.05\pi) = 40$

85. True. The period is $2\pi/(200\pi) = 1/100$ seconds. Thus, the function executes 100 cycles in 1 second.

86. False. If $\theta = \pi/2, 3\pi/2, 5\pi/2 \ldots$, then $\theta - \pi/2 = 0, \pi, 2\pi \ldots$, and the tangent is defined (it is zero) at these values.

87. False. When $x < 0$, we have $\sin|x| = \sin(-x) = -\sin x \neq \sin x$.

88. False. When $\pi < x < 2\pi$, we have $\sin|x| = \sin x < 0$ but $|\sin x| > 0$.

89. False. When $\pi/2 < x < 3\pi/2$, we have $\cos|x| = \cos x < 0$ but $|\cos x| > 0$.

90. True. Since $\cos(-x) = \cos x$, $\cos|x| = \cos x$.

91. False. For example, $\sin(0) \neq \sin((2\pi)^2)$, since $\sin(0) = 0$ but $\sin((2\pi)^2) = 0.98$.

92. True. Since $\sin(\theta + 2\pi) = \sin\theta$ for all θ, we have $g(\theta + 2\pi) = e^{\sin(\theta+2\pi)} = e^{\sin\theta} = g(\theta)$ for all θ.

93. False. A counterexample is given by $f(x) = \sin x$, which has period 2π, and $g(x) = x^2$. The graph of $f(g(x)) = \sin(x^2)$ in Figure 1.71 is not periodic with period 2π.

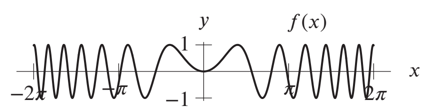


Figure 1.71

94. True. If $g(x)$ has period k, then $g(x + k) = g(x)$. Thus we have

$$f(g(x + k)) = f(g(x))$$

which shows that $f(g(x))$ is periodic with period k.

95. True, since $|\sin(-x)| = |-\sin x| = \sin x$.

96. False. If $t < -\pi/2$ or $t > \pi/2$, then $\sin^{-1}(\sin t) \neq t$.
Note, however, that $\sin^{-1}(\sin t) = t$ for $-\pi/2 \leq t \leq \pi/2$.

97. True. Since $\sin(t + 2\pi) = \sin t$, we have

$$f(t + 2\pi) = \sin^{-1}(\sin(t + 2\pi)) = \sin^{-1}(\sin t) = f(t).$$

To see that this is the smallest interval on which we have a complete cycle, notice that since $f(t) = 0$ only for $t = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, the only possible period shorter than 2π would be π. But $f(\pi/2)) = \pi/2$ and $f(\pi/2 + \pi) = -\pi/2$. Thus, π is not a period.

Solutions for Section 1.6

Exercises

1. As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to -\infty$.

2. As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to 0$.

3. Since $f(x)$ is an even power function with a negative leading coefficient, it follows that $f(x) \to -\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

4. Since $f(x)$ is an odd power function with a positive leading coefficient, it follows that $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

5. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of $5x^4$. Thus, as $x \to \pm\infty$, we see that $f(x) \to +\infty$.

6. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of $-5x^3$. Thus, as $x \to +\infty$, we see that $f(x) \to -\infty$ and as $x \to -\infty$, we see that $f(x) \to +\infty$.

7. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in the numerator and denominator. Thus, as $x \to \pm\infty$, we see that $f(x)$ behaves like $\dfrac{3x^2}{x^2} = 3$. We have $f(x) \to 3$ as $x \to \pm\infty$.

8. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in the numerator and denominator. Thus, as $x \to \pm\infty$, we see that $f(x)$ behaves like $\dfrac{-3x^3}{2x^3} = -3/2$. We have $f(x) \to -3/2$ as $x \to \pm\infty$.

9. As $x \to \pm\infty$, we see that $3x^{-4}$ gets closer and closer to 0, so $f(x) \to 0$ as $x \to \pm\infty$.

10. As $x \to +\infty$, we have $f(x) \to +\infty$. As $x \to -\infty$, we have $f(x) \to 0$.

11. The power function with the higher power dominates as $x \to \infty$, so $0.2x^5$ is larger.

12. An exponential growth function always dominates a power function as $x \to \infty$, so $10e^{0.1x}$ is larger.

13. An exponential growth function always dominates a power function as $x \to \infty$, so 1.05^x is larger.

14. The lower-power terms in a polynomial become insignificant as $x \to \infty$, so we are comparing $2x^4$ to the leading term $10x^3$. In comparing two power functions, the higher power dominates as $x \to \infty$, so $2x^4$ is larger.

15. The lower-power terms in a polynomial become insignificant as $x \to \infty$, so we are comparing the leading term $20x^4$ to the leading term $3x^5$. In comparing two power functions, the higher power dominates as $x \to \infty$, so the polynomial with leading term $3x^5$ is larger. As $x \to \infty$, we see that $25 - 40x^2 + x^3 + 3x^5$ is larger.

16. A power function with positive exponent dominates a log function, so as $x \to \infty$, we see that $\sqrt{x}$ is larger.

17. (I) **(a)** Minimum degree is 3 because graph turns around twice.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(II) **(a)** Minimum degree is 4 because graph turns around three times.
(b) Leading coefficient is positive because $y \to \infty$ as $x \to \infty$.
(III) **(a)** Minimum degree is 4 because graph turns around three times.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(IV) **(a)** Minimum degree is 5 because graph turns around four times.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(V) **(a)** Minimum degree is 5 because graph turns around four times.
(b) Leading coefficient is positive because $y \to \infty$ as $x \to \infty$.

18. (a) From the x-intercepts, we know the equation has the form

$$y = k(x+2)(x-1)(x-5).$$

Since $y = 2$ when $x = 0$,

$$\begin{aligned} 2 &= k(2)(-1)(-5) = k \cdot 10 \\ k &= \frac{1}{5}. \end{aligned}$$

Thus we have

$$y = \frac{1}{5}(x+2)(x-1)(x-5).$$

19. (a) Because our cubic has a root at 2 and a double root at -2, it has the form

$$y = k(x+2)(x+2)(x-2).$$

Since $y = 4$ when $x = 0$,

$$\begin{aligned} 4 &= k(2)(2)(-2) = -8k, \\ k &= -\frac{1}{2}. \end{aligned}$$

Thus our equation is

$$y = -\frac{1}{2}(x+2)^2(x-2).$$

20. $f(x) = k(x+3)(x-1)(x-4) = k(x^3 - 2x^2 - 11x + 12)$, where $k < 0$. ($k \approx -\frac{1}{6}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

21. $f(x) = kx(x+3)(x-4) = k(x^3 - x^2 - 12x)$, where $k < 0$. ($k \approx -\frac{2}{9}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

22. $f(x) = k(x+2)(x-1)(x-3)(x-5) = k(x^4 - 7x^3 + 5x^2 + 31x - 30)$, where $k > 0$. ($k \approx \frac{1}{15}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

23. $f(x) = k(x+2)(x-2)^2(x-5) = k(x^4 - 7x^3 + 6x^2 + 28x - 40)$, where $k < 0$. ($k \approx -\frac{1}{15}$ if the scales are equal; otherwise one can't tell how large k is.)

24. There are only two functions, h and p, which can be put in the form $y = Cb^x$, where C and b are constants:

$$p(x) = \frac{a^3 b^x}{c} = (a^3/c)b^x, \quad \text{where } C = a^3/c \text{ since } a, c \text{ are constants.}$$

$$h(x) = -\frac{1}{5^{x-2}} = -5^{-(x-2)} = -5^{-x+2} = (-25)5^{-x}.$$

Thus, h and p are the only exponential functions.

25. There is only one function, r, who can be put in the form $y = Ax^2 + Bx + C$:

$$r(x) = -x + b - \sqrt{cx^4} = -\sqrt{c}\,x^2 - x + b, \quad \text{where } A = -\sqrt{c}, \text{ since } c \text{ is a constant.}$$

Thus, r is the only quadratic function.

26. There is only one function, q, which is linear. Function q is a constant linear function whose vertical intercept is the constant ab^2/c, since a, b and c are constants.

27. Since a and b are constants, so is a/b, so $f(x)$, which can be rewritten as

$$f(x) = \frac{a}{b}x + c,$$

is a polynomial function. Also, $f(x)$ is a rational function since it can be written as a quotient of polynomials with numerator $f(x)$ and denominator 1. On the other hand, $f(x)$ is not a power function since it is the sum of two terms with different powers of x, and as there are no terms with x as an exponent, it cannot be an exponential function.

Thus, $f(x)$ falls into families (III), and (IV).

28. By combining the two terms of $g(x)$ over a common denominator, $g(x)$ can be rewritten as

$$g(x) = ax^2 + \frac{b}{x^2} = \frac{ax^4}{x^2} + \frac{b}{x^2} = \frac{ax^4 + b}{x^2}.$$

Therefore, since $g(x)$ can be written as a quotient of polynomials, it is a rational function. However, $g(x)$ is not a power function since it is the sum of two terms with different powers of x, it is not a polynomial since it has terms with negative powers of x, and as there are no terms with x as an exponent, it cannot be an exponential function.

Thus, $g(x)$ falls into family (IV) only.

29. Since b and c are constants, so is b/c^a. In addition, since a is a positive integer, $h(x)$, which can be rewritten as

$$h(x) = b\left(\frac{x}{c}\right)^a = \frac{b}{c^a}x^a,$$

is a power function. It is also a polynomial, since a polynomial is a sum of power functions with positive integer exponents, and it is a rational function, since any polynomial can be viewed as a rational function with denominator 1. However, $h(x)$ is not an exponential function.

Thus, $h(x)$ falls into families (II), (III) and (IV).

30. Since a is constant, $k(x)$ is a power function. Since a is a positive integer and a polynomial is a sum of power functions with non-negative integer powers, $k(x)$ is a polynomial. Also, $k(x)$ is a rational function as it is the quotient of the two polynomials ax and 1. On the other hand, $k(x)$ is not exponential since there are no terms with x as an exponent.

Thus, $k(x)$ falls into families (II), (III), and (IV).

31. Since a and b are constants, so is $a+b$, so $j(x)$, which can be written in the form

$$j(x) = ax^{-1} + \frac{b}{x} = ax^{-1} + bx^{-1} = (a+b)x^{-1},$$

is a power function. We can also write $j(x)$ as

$$j(x) = (a+b)x^{-1} = \frac{a+b}{x},$$

a quotient of polynomials, so it is also a rational function. The function $j(x)$ is not a polynomial as there are terms with negative powers of x, and $j(x)$ is not exponential since there are no terms with x as an exponent.

Thus, $j(x)$ falls into families (II), and (IV).

32. Since a, b, and c are all constants, so is $((a+b)/c)^2$, so $l(x)$, which can be written in the form

$$l(x) = \left(\frac{a+b}{c}\right)^{2x} = \left(\left(\frac{a+b}{c}\right)^2\right)^x,$$

is an exponential function. It is neither a polynomial, rational, nor power function since there is a variable in the exponent of one of the terms.

Thus, $l(x)$ falls into family (I).

33. IV, since expanding gives

$$y = x^3 - x.$$

34. III and VI, since expanding gives

$$y = x^4 - 2x^2 + 1 \qquad \text{and} \qquad y = x^4 - 4.$$

35. I, II, and V. Both I and II are the ratios of polynomials; V can be rewritten as the ratio of polynomials as:

$$y = \frac{x^2-1}{x}.$$

Since in each case, the denominator is not a constant, these functions cannot be thought of as polynomials.

36. II, since the numerator has two zeros, $x = \pm\sqrt{2}$.
III, since the polynomial factors to $y = -(x-1)^2(x+1)^2$, it has two zeros, $x = \pm 1$.
V, since we can combine terms, $y = \dfrac{x^2-1}{x}$, and the numerator has two zeros, $x = \pm 1$.
VI, since $y = (x^2-2)(x^2+2)$ has two distinct zeros, $x = \pm\sqrt{2}$, from the first factor. The second factor does not contribute any zeros.

37. V, since the denominator has exactly one zero, $x = 0$.

38. IV, since it factors to $y = x(x-1)(x+1)$.

39. II, since the denominator has two zeros, $x = \pm\sqrt{2}$.

40. I and II, since in each case as $x \to \infty$ or $x \to -\infty$, we have

$$y \to y = \frac{x^2}{x^2} = 1.$$

41. Since the graph of this function is only defined for positive values of x, this is the graph of a logarithmic function.

42. Since the graph of this function has only one horizontal asymptote, this is the graph of an exponential function.

43. Since the graph of this function has two horizontal asymptotes, it must be a rational function.

44. Since this function both increases and decreases over different parts of its domain, it must be a rational function.

Problems

45. Consider the end behavior of the graph; that is, as $x \to +\infty$ and $x \to -\infty$. The ends of a degree 5 polynomial are in Quadrants I and III if the leading coefficient is positive or in Quadrants II and IV if the leading coefficient is negative.

Thus, there must be at least one root. Since the degree is 5, there can be no more than 5 roots. Thus, there may be 1, 2, 3, 4, or 5 roots. Graphs showing these five possibilities are shown in Figure 1.72.

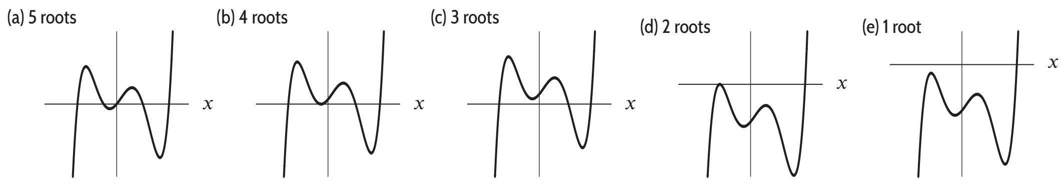


Figure 1.72

46. The graphs of both these functions will resemble that of x^3 on a large enough window. One way to tackle the problem is to graph them both (along with x^3 if you like) in successively larger windows until the graphs come together. In Figure 1.73, f, g and x^3 are graphed in four windows. In the largest of the four windows the graphs are indistinguishable, as required. Answers may vary.

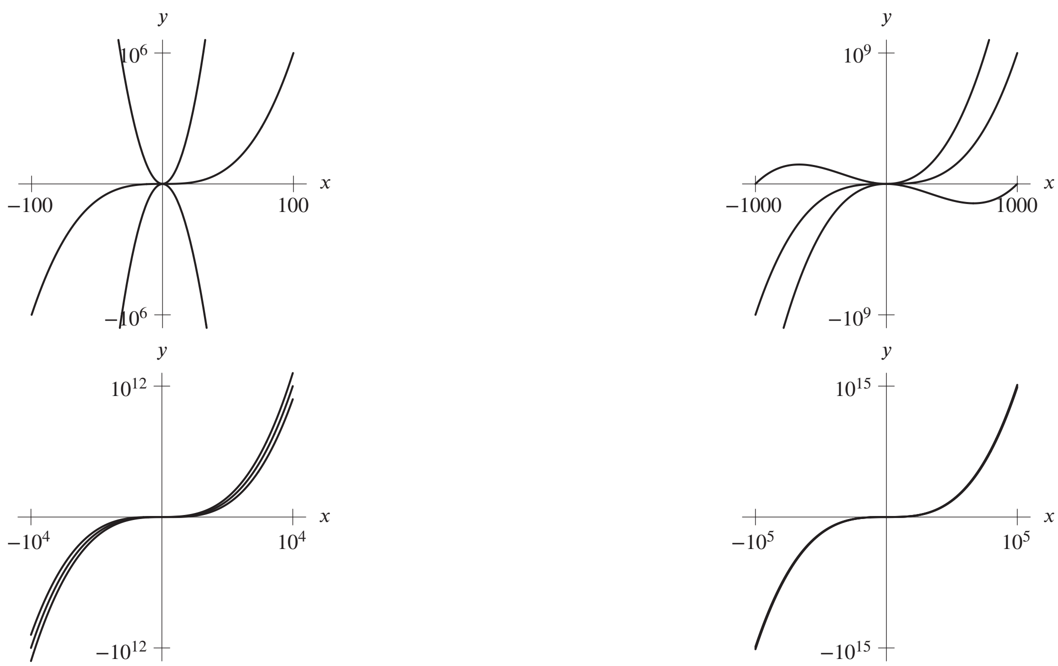


Figure 1.73

47. The function is a cubic polynomial with positive leading coefficient. Since the figure given in the text shows that the function turns around once, we know that the function has the shape shown in Figure 1.74. The function is below the x-axis for $x = 5$ in the given graph, and we know that it goes to $+\infty$ as $x \to +\infty$ because the leading coefficient is positive. Therefore, there are exactly three zeros. Two zeros are shown, and occur at approximately $x = -1$ and $x = 3$. The third zero must be to the right of $x = 10$ and so occurs for some $x > 10$.

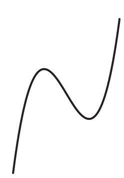


Figure 1.74

48. (a) (i) If (1, 1) is on the graph, we know that

$$1 = a(1)^2 + b(1) + c = a + b + c.$$

(ii) If (1, 1) is the vertex, then the axis of symmetry is $x = 1$, so

$$-\frac{b}{2a} = 1,$$

and thus

$$a = -\frac{b}{2}, \text{ so } b = -2a.$$

But to be the vertex, (1, 1) must also be on the graph, so we know that $a + b + c = 1$. Substituting $b = -2a$, we get $-a + c = 1$, which we can rewrite as $a = c - 1$, or $c = 1 + a$.

(iii) For (0, 6) to be on the graph, we must have $f(0) = 6$. But $f(0) = a(0^2) + b(0) + c = c$, so $c = 6$.

(b) To satisfy all the conditions, we must first, from (a)(iii), have $c = 6$. From (a)(ii), $a = c - 1$ so $a = 5$. Also from (a)(ii), $b = -2a$, so $b = -10$. Thus the completed equation is

$$y = f(x) = 5x^2 - 10x + 6,$$

which satisfies all the given conditions.

49. Let us represent the height by h. Since the volume is V, we have

$$x^2 h = V.$$

Solving for h gives

$$h = \frac{V}{x^2}.$$

The graph is in Figure 1.75. We are assuming V is a positive constant.

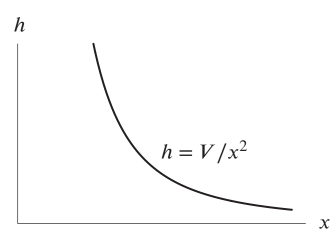


Figure 1.75

50. (a) Let the height of the can be h. Then

$$V = \pi r^2 h.$$

The surface area consists of the area of the ends (each is πr^2) and the curved sides (area $2\pi r h$), so

$$S = 2\pi r^2 + 2\pi r h.$$

Solving for h from the formula for V, we have

$$h = \frac{V}{\pi r^2}.$$

Substituting into the formula for S, we get

$$S = 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

(b) For large r, the $2V/r$ term becomes negligible, meaning $S \approx 2\pi r^2$, and thus $S \to \infty$ as $r \to \infty$.

(c) The graph is in Figure 1.76.

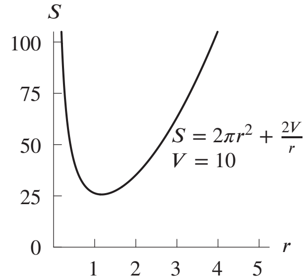


Figure 1.76

51. Substituting $w = 65$ and $h = 160$, we have

(a)
$$s = 0.01(65^{0.25})(160^{0.75}) = 1.3 \text{ m}^2.$$

(b) We substitute $s = 1.5$ and $h = 180$ and solve for w:
$$1.5 = 0.01w^{0.25}(180^{0.75}).$$
We have
$$w^{0.25} = \frac{1.5}{0.01(180^{0.75})} = 3.05.$$
Since $w^{0.25} = w^{1/4}$, we take the fourth power of both sides, giving
$$w = 86.8 \text{ kg}.$$

(c) We substitute $w = 70$ and solve for h in terms of s:
$$s = 0.01(70^{0.25})h^{0.75},$$
so
$$h^{0.75} = \frac{s}{0.01(70^{0.25})}.$$
Since $h^{0.75} = h^{3/4}$, we take the $4/3$ power of each side, giving
$$h = \left(\frac{s}{0.01(70^{0.25})}\right)^{4/3} = \frac{s^{4/3}}{(0.01^{4/3})(70^{1/3})}$$
so
$$h = 112.6s^{4/3}.$$

52. Let $D(v)$ be the stopping distance required by an Alpha Romeo as a function of its velocity. The assumption that stopping distance is proportional to the square of velocity is equivalent to the equation
$$D(v) = kv^2$$
where k is a constant of proportionality. To determine the value of k, we use the fact that $D(70) = 150$.
$$D(70) = k(70)^2 = 150.$$
Thus,
$$k = \frac{150}{70^2} \approx 0.0306.$$
It follows that
$$D(35) = \left(\frac{150}{70^2}\right)(35)^2 = \frac{150}{4} = 37.5 \text{ ft}$$
and
$$D(160) = \left(\frac{150}{70^2}\right)(140)^2 = 150 \cdot 4 = 600 \text{ ft}.$$
Thus, at half the speed it requires one fourth the distance, whereas at twice the speed it requires four times the distance, as we would expect from the equation. (We could in fact have figured it out that way, without solving for k explicitly.)

53. (a) Since the rate R varies directly with the fourth power of the radius r, we have the formula

$$R = kr^4$$

where k is a constant.

(b) Given $R = 400$ for $r = 3$, we can determine the constant k.

$$\begin{aligned} 400 &= k(3)^4 \\ 400 &= k(81) \\ k &= \frac{400}{81} \approx 4.93827. \end{aligned}$$

So the formula is

$$R = 4.93827r^4$$

(c) Evaluating the formula above at $r = 5$ yields

$$R = 4.93827(5)^4 = 3086.42\frac{\text{cm}^3}{\text{sec}}.$$

54. The pomegranate is at ground level when $f(t) = -16t^2 + 64t = -16t(t-4) = 0$, so when $t = 0$ or $t = 4$. At time $t = 0$ it is thrown, so it must hit the ground at $t = 4$ seconds. The symmetry of its path with respect to time may convince you that it reaches its maximum height after 2 seconds. Alternatively, we can think of the graph of $f(t) = -16t^2 + 64t = -16(t-2)^2 + 64$, which is a downward parabola with vertex (i.e., highest point) at $(2, 64)$. The maximum height is $f(2) = 64$ feet.

55. (a) The object starts at $t = 0$, when $s = v_0(0) - g(0)^2/2 = 0$. Thus it starts on the ground, with zero height.

(b) The object hits the ground when $s = 0$. This is satisfied at $t = 0$, before it has left the ground, and at some later time t that we must solve for.

$$0 = v_0 t - gt^2/2 = t\left(v_0 - gt/2\right)$$

Thus $s = 0$ when $t = 0$ and when $v_0 - gt/2 = 0$, i.e., when $t = 2v_0/g$. The starting time is $t = 0$, so it must hit the ground at time $t = 2v_0/g$.

(c) The object reaches its maximum height halfway between when it is released and when it hits the ground, or at

$$t = (2v_0/g)/2 = v_0/g.$$

(d) Since we know the time at which the object reaches its maximum height, to find the height it actually reaches we just use the given formula, which tells us s at any given t. Substituting $t = v_0/g$,

$$\begin{aligned} s &= v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0^2}{g^2}\right) = \frac{v_0^2}{g} - \frac{v_0^2}{2g} \\ &= \frac{2v_0^2 - v_0^2}{2g} = \frac{v_0^2}{2g}. \end{aligned}$$

56. (a) $R(P) = kP(L-P)$, where k is a positive constant.

(b) A possible graph is in Figure 1.77.

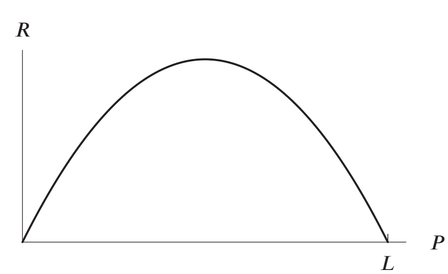


Figure 1.77

57. Yes. The graph of this power function is increasing and concave down for positive mass M. See Figure 1.78. This function could represent the length of a plant species that stretches relatively less per unit mass increase as the plant gets larger.

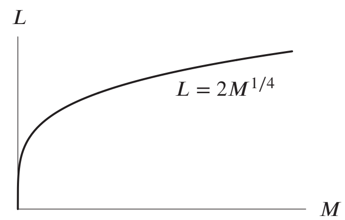


Figure 1.78

58. No. The graph of this polynomial function is increasing but concave up for positive mass M. See Figure 1.79. A plant growing according to this function would stretch relatively more (not less) per unit mass increase as the plant gets larger.

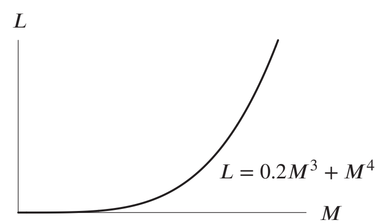


Figure 1.79

59. No. The graph of this power function is decreasing for positive mass M. See Figure 1.80. A plant growing according to this function would actually shrink in length (rather than stretch) as its mass increases.

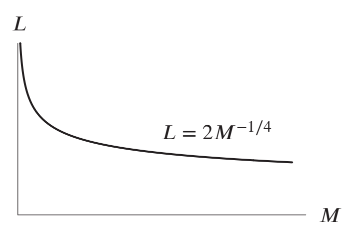


Figure 1.80

60. Yes. The graph of this rational function is increasing and concave down. See Figure 1.81. A plant species growing according to this function stretches relatively less per unit mass increase as the plant gets larger.

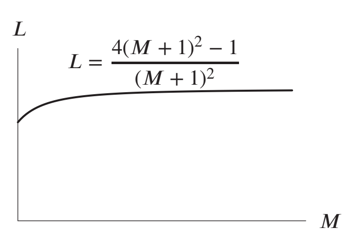


Figure 1.81

61. No. The graph of this rational function is decreasing for positive mass M. See Figure 1.82. A plant growing according to this function would actually shrink in length (rather than stretch) as its mass increases.

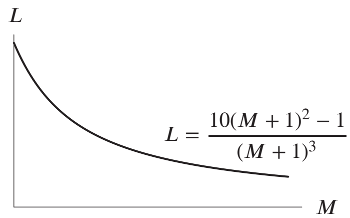


Figure 1.82

62. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that

$$f(x) \to \frac{5x}{2x} = \frac{5}{2}.$$

There is a horizontal asymptote at $y = 5/2$.
To find the vertical asymptotes, we set the denominator equal to zero. When $2x + 3 = 0$, we have $x = -3/2$ so there is a vertical asymptote at $x = -3/2$.

63. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that

$$f(x) \to \frac{x^2}{x^2} = 1.$$

There is a horizontal asymptote at $y = 1$.
To find the vertical asymptotes, we set the denominator equal to zero. When $x^2 - 4 = 0$, we have $x = \pm 2$ so there are vertical asymptotes at $x = -2$ and at $x = 2$.

64. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that

$$f(x) \to \frac{5x^3}{x^3} = 5.$$

There is a horizontal asymptote at $y = 5$.
To find the vertical asymptotes, we set the denominator equal to zero. When $x^3 - 27 = 0$, we have $x = 3$ so there is a vertical asymptote at $x = 3$.

65. (a) A polynomial has the same end behavior as its leading term, so this polynomial behaves as $-5x^4$ globally. Thus we have:
$f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to -\infty$ as $x \to +\infty$.
(b) Polynomials behave globally as their leading term, so this rational function behaves globally as $(3x^2)/(2x^2)$, or $3/2$. Thus we have:
$f(x) \to 3/2$ as $x \to -\infty$, and $f(x) \to 3/2$ as $x \to +\infty$.
(c) We see from a graph of $y = e^x$ that
$f(x) \to 0$ as $x \to -\infty$, and $f(x) \to +\infty$ as $x \to +\infty$.

66. $g(x) = 2x^2$, $h(x) = x^2 + k$ for any $k > 0$. Notice that the graph is symmetric about the y-axis and $\lim_{x\to\infty} f(x) = 2$.

67. We use the fact that at a constant speed, Time = Distance/Speed. Thus,

$$\begin{aligned}\text{Total time} &= \text{Time running} + \text{Time walking}\\ &= \frac{3}{x} + \frac{6}{x-2}.\end{aligned}$$

Horizontal asymptote: x-axis.
Vertical asymptote: $x = 0$ and $x = 2$.

68. (a) II and III because in both cases, the numerator and denominator each have x^2 as the highest power, with coefficient $= 1$. Therefore,

$$y \to \frac{x^2}{x^2} = 1 \quad \text{as } x \to \pm\infty.$$

(b) I, since

$$y \to \frac{x}{x^2} = 0 \quad \text{as } x \to \pm\infty.$$

(c) II and III, since replacing x by $-x$ leaves the graph of the function unchanged.

(d) None

(e) III, since the denominator is zero and $f(x)$ tends to $\pm\infty$ when $x = \pm 1$.

69. $h(t)$ cannot be of the form ct^2 or kt^3 since $h(0.0) = 2.04$. Therefore $h(t)$ must be the exponential, and we see that the ratio of successive values of h is approximately 1.5. Therefore $h(t) = 2.04(1.5)^t$. If $g(t) = ct^2$, then $c = 3$ since $g(1.0) = 3.00$. However, $g(2.0) = 24.00 \neq 3 \cdot 2^2$. Therefore $g(t) = kt^3$, and using $g(1.0) = 3.00$, we obtain $g(t) = 3t^3$. Thus $f(t) = ct^2$, and since $f(2.0) = 4.40$, we have $f(t) = 1.1t^2$.

70. The graphs are shown in Figure 1.83.

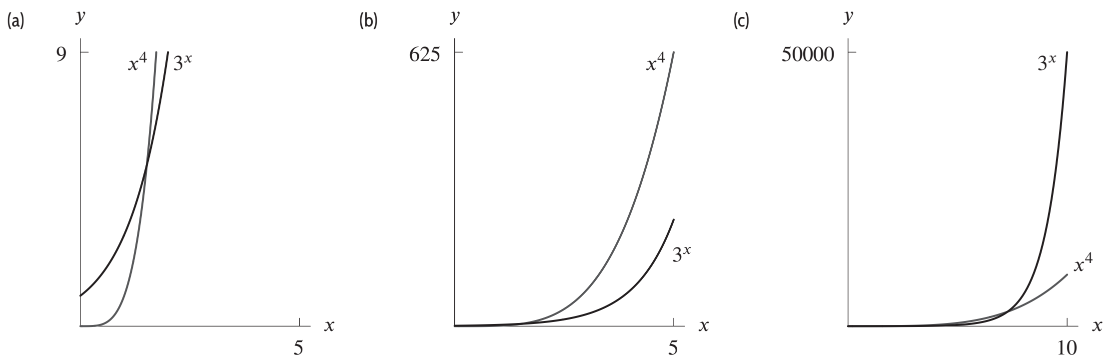


Figure 1.83

71. Since the parabola opens upward, we must have $a > 0$. To determine a relationship between x and y at the point of intersection P, we eliminate a from the parabola and circle equations. Since $y = x^2/a$, we have $a = x^2/y$. Putting this into the circle equation gives $x^2 + y^2 = 2x^4/y^2$. Rewrite this as

$$\begin{aligned} x^2y^2 + y^4 &= 2x^4 \\ y^4 + x^2y^2 - 2x^4 &= 0 \\ (y^2 + 2x^2)(y^2 - x^2) &= 0. \end{aligned}$$

This means $x^2 = y^2$ (since y^2 cannot equal $-2x^2$). Thus $x = y$ since P is in the first quadrant. So P moves out along the line $y = x$ through the origin.

72. (a) The graph is shown in Figure 1.84. The graph represented by the exact formula has a vertical asymptote where the denominator is undefined. This happens when

$$1 - \frac{v^2}{c^2} = 0, \text{ or at } v^2 = c^2.$$

Since $v > 0$, the graph of the exact formula has a vertical asymptote where

$$v = c = 3 \cdot 10^8 \text{m/sec}.$$

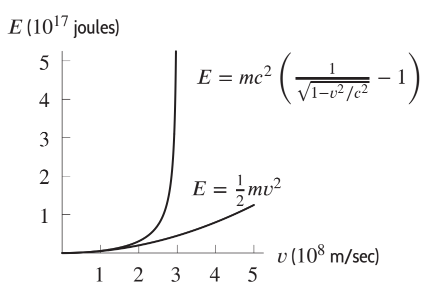


Figure 1.84

(b) The first formula does not give a good approximation to the exact formula when the graphs are not close together. This happens for $v > 1.5 \cdot 10^8$ m/sec. For $v < 1.5 \cdot 10^8$ m/sec, the graphs look close together. However, the vertical scale we are using is so large and the graphs are so close to the v-axis that a more careful analysis should be made. We should zoom in and redraw the graph.

73. Since $z = \ln x$, we have $x = e^z$. Hence

$$y = 100x^{-0.2} = 100(e^z)^{-0.2} = 100e^{-0.2z}.$$

Strengthen Your Understanding

74. The graph of a polynomial of degree 5 cuts the horizontal axis at most five times, but it could be fewer. For example, $f(x) = x^5$ cuts the x-axis only once.

75. A fourth degree polynomial tends to negative infinity if the leading coefficient (coefficient of the x^4 term) is negative. For example, $f(x) = -x^4$ tends to negative infinity as $x \to \infty$.

76. A rational function has no vertical asymptotes if the denominator has no zeros. For example:

$$f(x) = \frac{1}{x^2+1}.$$

77. As seen from the graphs of the functions in Figure 1.85, this inequality only holds for $x > 1$. The functions cross at the point $(1, 1)$ and the inequality is reversed for $0 < x < 1$.

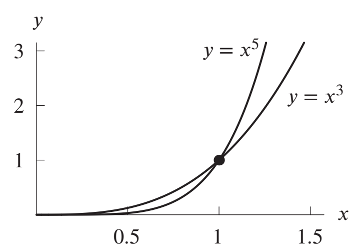


Figure 1.85

78. The rational function $f(x) = (x^3 + 1)/x$ has no horizontal asymptotes. To see this, observe that

$$y = f(x) = \frac{x^3+1}{x} \approx \frac{x^3}{x} = x^2$$

for large x. Thus, $y \to \infty$ as $x \to \pm\infty$.

79. Let $f(x) = x/(x^2 + 1)$. Since

$$y = \frac{x}{x^2+1} \approx \frac{x}{x^2} = \frac{1}{x} \quad \text{for large } x,$$

and $1/x \to 0$ as $x \to \infty$, the function $f(x)$ has horizontal asymptote $y = 0$. We have $f(0) = 0$, so $f(x)$ crosses this asymptote at $x = 0$.

80. One possibility is $p(x) = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$.

81. A possible function is

$$f(x) = \frac{3x}{x-10}.$$

82. One possibility is

$$f(x) = \frac{1}{x^2+1}.$$

83. Let $f(x)=\dfrac{1}{x+7\pi}$. Other answers are possible.

84. Let $f(x)=\dfrac{1}{(x-1)(x-2)(x-3)\cdots(x-16)(x-17)}$. This function has an asymptote corresponding to every factor in the denominator. Other answers are possible.

85. The function $f(x)=\dfrac{x-1}{x-2}$ has $y=1$ as the horizontal asymptote and $x=2$ as the vertical asymptote. These lines cross at the point $(2,1)$. Other answers are possible.

86. False. The polynomial $f(x)=x^2+1$, with degree 2, has no real zeros.

87. True. If the degree of the polynomial, $p(x)$, is n, then the leading term is a_nx^n with $a_n\neq 0$.

If n is odd and a_n is positive, $p(x)$ tends toward ∞ as $x\to\infty$ and $p(x)$ tends toward $-\infty$ as $x\to-\infty$. Since the graph of $p(x)$ has no breaks in it, the graph must cross the x-axis at least once.

If n is odd and a_n is negative, a similar argument applies, with the signs reversed, but leading to the same conclusion.

88. False. Suppose $f(x)=g(x)=x^2$. Then the composition is quartic (fourth degree), not quadratic:

$$f(g(x))=f(x^2)=(x^2)^2=x^4.$$

89. True. If we factor and simplify the rational expression, we see that

$$f(x)=\frac{5x(x+1)(x-1)}{x(x+1)}=5x-5,\ \text{for } x\neq -1,0.$$

Hence, over $x>0$ the graph of the f is the same at the graph of the linear function $y=5x-5$.

90. (a), (c), (d), (e), (b). Notice that $f(x)$ and $h(x)$ are decreasing functions, with $f(x)$ being negative. Power functions grow slower than exponential growth functions, so $k(x)$ is next. Now order the remaining exponential functions, where functions with larger bases grow faster.

Additional Problems (online only)

91. (a) (I) For linear functions we have

$$f(x+2)-f(x)=b+m(x+2)-(b+mx)=2m.$$

The difference is constant, because it does not depend on x.

(b) (III) For exponential functions we have

$$\begin{aligned}\frac{f(x+2)}{f(x)}&=\frac{ae^{k(x+2)}}{ae^{kx}}\\&=\frac{ae^{kx}e^{2k}}{ae^{kx}}=e^{2k}.\end{aligned}$$

The quotient is constant, because it does not depend on x.

(c) (IV) For power functions we have

$$\begin{aligned}\frac{f(2x)}{f(x)}&=\frac{a(2x)^n}{ax^n}\\&=\frac{a2^nx^n}{ax^n}=2^n.\end{aligned}$$

The quotient is constant, because it does not depend on x.

(d) (II) For logarithmic functions we have

$$\begin{aligned}f(2x)-f(x)&=(a+b\ln(2x))-(a+b\ln x)\\&=(a+b\ln 2+b\ln x)-(a+b\ln x)=b\ln 2.\end{aligned}$$

The difference is constant, because it does not depend on x.

92. Properties of logs give

$$\ln(kx^p) = \ln k + \ln x^p = \ln k + p\ln x.$$

Therefore

$$\begin{aligned}\frac{\ln f(x_2) - \ln f(x_1)}{\ln x_2 - \ln x_1} &= \frac{\ln(kx_2^p) - \ln(kx_1^p)}{\ln x_2 - \ln x_1}\\ &= \frac{\ln k + p\ln x_2 - \ln k - p\ln x_1}{\ln x_2 - \ln x_1}\\ &= \frac{p(\ln x_2 - \ln x_1)}{\ln x_2 - \ln x_1} = p.\end{aligned}$$

Solutions for Section 1.7

Exercises

1. (a) The function f has jumps in the graph at $x = -1$ and at $x = 1$. Therefore, f is not continuous at $x = -1$ and at $x = 1$.
(b) The domain of f is $-3 \le x \le 3$, and by part (a), we know that f is not continuous at $x = -1$ and at $x = 1$. Therefore, f is continuous on $-3 < x < -1$, $-1 < x < 1$, and $1 < x < 3$.

2. (a) The function f has a break in the graph at $x = 1$ and jumps in the graph at $x = 2$ and at $x = 4$. Therefore, f is not continuous at $x = 1, 2$, and 4.
(b) The domain of f is $0 \le x \le 6$, and by part (a), we know that f is not continuous at $x = 1, 2$ and at 4. Therefore, f is continuous on $0 < x < 1$, $1 < x < 2$, $2 < x < 4$, and $4 < x < 6$.

3. (a) As x approaches -3 from either side, the values of $f(x)$ get closer and closer to 1, so the limit appears to be about 1.
(b) As x approaches -2 from either side, the values of $f(x)$ get closer and closer to 2, so the limit appears to be about 2.
(c) As x approaches -1, the values of $f(x)$ get closer and closer to 3 on one side of $x = -1$ and get closer and closer to 2 on the other side of $x = -1$. Thus the limit does not exist.
(d) As x approaches 0 from either side, the values of $f(x)$ get closer and closer to 1, so the limit appears to be about 1.
(e) As x approaches 1 from either side, the values of $f(x)$ get closer and closer to 0, so the limit appears to be about 0.
(f) As x approaches 3 from either side, the values of $f(x)$ get closer and closer to 2. (Recall that to find a limit, we are interested in what happens to the function near x but not at x.) The limit appears to be about 2.

4. (a) Since the graph has a jump at $x = -1$, $f(x)$ is not continuous at $x = -1$. Since the graph has a hole at $x = 2$, $f(x)$ is not continuous at $x = 2$.
(b) As x approaches -1 from either side, the values of $f(x)$ get closer and closer to 2 on one side of $x = -1$ and get closer and closer to 5 on the other side of $x = -1$. Thus $\lim_{x\to -1} f(x)$ does not exist. As x approaches 2 from either side, the values of $f(x)$ get closer and closer to -3, so the limit appears to be about -3.

5. (a) Since the graph has a hole at $x = -2$, $f(x)$ is not continuous at $x = -2$. Since the graph has an asymptote at $x = 3$, $f(x)$ is not continuous at $x = 3$.
(b) As x approaches -2 from either side, the values of $f(x)$ get closer and closer to -3, so the limit appears to be about -3. As x approaches 3 from either side, the values of $f(x)$ approach ∞. Thus $\lim_{x\to 3} f(x)$ does not exist.

6. Values of $f(x)$ appear to approach 2 as x gets close to 0 from both sides, so $\lim_{x\to 0} f(x)$ appears to be 2.

We do not know for sure the limit is 2 because for values of x close to 0 different from those in the table, the values of $f(x)$ might not approach 2. For example, they might not approach a limit at all, or they might approach, for example, 2.00001 or 1.9999.

7. Values of $g(t)$ appear to approach 1 as t gets close to 3 from both sides, so $\lim_{t\to 3} g(t)$ appears to be 1.

We do not know for sure the limit is 1 because for values of t close to 3 different from those in the table, the values of $g(t)$ might not approach 1. For example, they might not approach a limit at all, or they might approach, for example, 1.00001 or 0.9999.

8. (a) Substituting x-values into $f(x)$ gives:

x	-0.1	-0.01	0.01	0.1
$f(x)$	4.794	4.998	4.998	4.794

(b) Since the values of $f(x)$ appear to get closer and closer to 5 as x gets closer and closer to 0, we estimate

$$\lim_{x\to 0} \frac{\sin(5x)}{x} = 5.$$

Notice that $f(x) = \dfrac{\sin(5x)}{x}$ is undefined at $x = 0$.

9. (a) Substituting x-values into $f(x)$ gives:

x	-0.1	-0.01	0.01	0.1
$f(x)$	2.592	2.955	3.045	3.499

(b) Since the values of $f(x)$ appear to get closer and closer to 3 as x gets closer and closer to 0, we estimate

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} = 3.$$

Notice that $f(x) = \dfrac{e^{3x}-1}{x}$ is undefined at $x = 0$.

10. We evaluate $f(x) = 5 + \ln x$ for values of x closer and closer to 1.

x	0.9	0.99	0.999	1.001	1.01	1.1
$5 + \ln x$	4.895	4.990	4.999	5.001	5.010	5.095

From the table, we see that $f(x)$ gets closer and closer to 5, so we estimate

$$\lim_{x\to 1}(5 + \ln x) = 5.$$

11. Yes, because $x - 2$ is not zero on this interval.

12. No, because $x - 2 = 0$ at $x = 2$.

13. Yes, because $2x - 5$ is positive for $3 \le x \le 4$.

14. No, because $2x + x^{-1}$ is undefined at $x = 0$.

15. No, because $\cos(\pi/2) = 0$.

16. Yes, because $\cos\theta$ is not zero on this interval.

17. (a) At $x = 1$, on the line $y = x$, we have $y = 1$. At $x = 1$, on the parabola $y = x^2$, we have $y = 1$. Thus, $f(x)$ is continuous. See Figure 1.86.

(b) At $x = 3$, on the line $y = x$, we have $y = 3$. At $x = 3$, on the parabola $y = x^2$, we have $y = 9$. Thus, $g(x)$ is not continuous. See Figure 1.87.

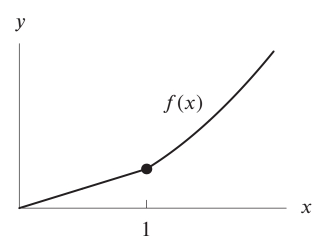


Figure 1.86

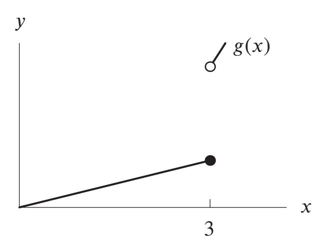


Figure 1.87

18. (a) Since the graph of f is comprised of portions of either linear or quadratic functions, f is continuous everywhere except possibly at $x = 3$ and $x = 5$, where the formula for the function f changes.

Since the graph of $4 - x$ goes through the point $(3, 1)$ but the graph of $x^2 - 8x + 17$ goes through the point $(3, 2)$, there is a jump in the graph of f at $x = 3$, meaning that f is not continuous at $x = 3$.

On the other hand, the graph of $x^2 - 8x + 17$ and $12 - 2x$ both go through the point $(5, 2)$, so f is continuous at $x = 5$.

Thus, $x = 3$ is the only value of x at which f is not continuous.

(b) By part (a), we know that f is not continuous at $x = 3$. Therefore, f is continuous on $0 < x < 3$ and $3 < x < 6$.

19. We have that $f(0) = -1 < 0$ and $f(1) = 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

20. We have that $f(0) = 1 > 0$ and $f(1) = e - 3 < 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

21. We have that $f(0) = -1 < 0$ and $f(1) = 1 - \cos 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

22. Since f is not continuous at $x = 0$, we consider instead the smaller interval $[0.01, 1]$. We have that $f(0.01) = 2^{0.01} - 100 < 0$ and $f(1) = 2 - 1/1 = 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0.01, 1]$, and hence in $[0, 1]$, such that $f(c) = k = 0$.

23. Evaluating $\dfrac{x^2-4}{x-2}$ at $x = 2$ gives us $0/0$, so we see if we can rewrite the function using algebra. We have

$$\frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2}.$$

Since $x \neq 2$ in the limit, we can cancel the common factor $x - 2$ to see

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

24. Evaluating $\dfrac{x^2-9}{x+3}$ at $x = -3$ gives us $0/0$, so we see if we can rewrite the function using algebra. We have

$$\frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3}.$$

Since $x \neq -3$ in the limit, we can cancel the common factor $x + 3$ to see

$$\lim_{x\to -3} \frac{x^2-9}{x+3} = \lim_{x\to -3} \frac{(x+3)(x-3)}{x+3} = \lim_{x\to -3}(x-3) = -6.$$

25. Evaluating $\dfrac{x^2+4x-5}{x-1}$ at $x = 1$ gives us $0/0$, so we see if we can rewrite the function using algebra. We have

$$\frac{x^2+4x-5}{x-1} = \frac{(x+5)(x-1)}{x-1}.$$

Since $x \neq 1$ in the limit, we can cancel the common factor $x - 1$ to see

$$\lim_{x\to 1} \frac{x^2+4x-5}{x-1} = \lim_{x\to 1} \frac{(x+5)(x-1)}{x-1} = \lim_{x\to 1}(x+5) = 6.$$

26. Evaluating $\dfrac{x^2-4x+3}{x^2+3x-4}$ at $x = 1$ gives us $0/0$, so we see if we can rewrite the function using algebra. We have

$$\frac{x^2-4x+3}{x^2+3x-4} = \frac{(x-1)(x-3)}{(x+4)(x-1)}.$$

Since $x \neq 1$ in the limit, we can cancel the common factor $x - 1$ to see

$$\lim_{x\to 1} \frac{x^2-4x+3}{x^2+3x-4} = \lim_{x\to 1} \frac{(x-1)(x-3)}{(x+4)(x-1)} = \lim_{x\to 1} \frac{x-3}{x+4} = \frac{-2}{5}.$$

27. The expression $\dfrac{x^2+4}{x+8}$ is continuous everywhere on its domain (that is, at every point except $x = -8$.) In particular, it is continuous at $x = 1$ so we find the limit by evaluating the expression at $x = 1$. We have

$$\lim_{x\to 1} \frac{x^2+4}{x+8} = \frac{5}{9}.$$

28. Evaluating the expression at $h = 0$, we arrive at $0/0$. Rewriting the expression using algebra gives

$$\begin{aligned}
\lim_{h\to 0} \frac{(5+h)^2 - 5^2}{h} &= \lim_{h\to 0} \frac{(25 + 10h + h^2) - 25}{h} \\
&= \lim_{h\to 0} \frac{10h + h^2}{h} \\
&= \lim_{h\to 0} \frac{h(10+h)}{h} \\
&\text{and since } h \neq 0 \text{, canceling the common factor of } h \text{, we have} \\
&= \lim_{h\to 0} (10 + h) \\
&= 10.
\end{aligned}$$

29. Since $kx + 10$ is continuous at $x = 5$, the limit can be found by substituting $x = 5$. Thus,

$$\begin{aligned}
k(5) + 10 &= 20 \\
k &= 2.
\end{aligned}$$

30. Since $(x+6)(x-k)/(x^2+x)$ is continuous at $x = 2$, the limit can be found by substituting $x = 2$. Thus,

$$\begin{aligned}
\frac{(2+6)(2-k)}{2^2+2} &= 4 \\
k &= -1.
\end{aligned}$$

31. The value of y on the line $y = kx$ at $x = 3$ is $y = 3k$. To make $f(x)$ continuous, we need

$$3k = 5 \quad \text{so} \quad k = \frac{5}{3}.$$

See Figure 1.88.

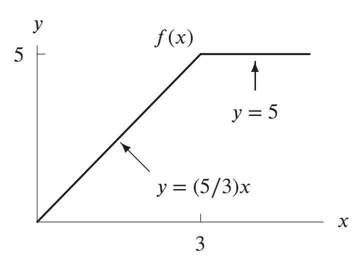


Figure 1.88

32. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $3x^2$ takes the value $3(2^2) = 12$ at $x = 2$, we choose k so that the graph of kx goes through the point $(2, 12)$. Thus $k = 6$.

33. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.

Since $x + 3$ takes the value $1 + 3 = 4$ at $x = 1$, we choose k so that the graph of kx goes through the point $(1, 4)$. Thus $k = 4$.

34. If the graphs of $y = t + k$ and $y = kt$ meet at $t = 5$, we have

$$\begin{aligned}
5 + k &= 5k \\
k &= 5/4.
\end{aligned}$$

See Figure 1.89.

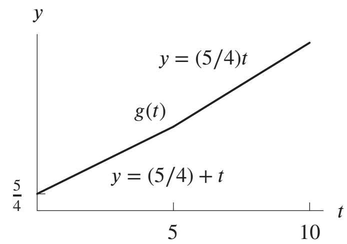


Figure 1.89

Problems

35. (a) Even if the car stops to refuel, the amount of fuel in the tank changes smoothly, so the fuel in the tank is a continuous function; the quantity of fuel cannot suddenly change from one value to another.
(b) Whenever a student joins or leaves the class the number jumps up or down immediately by 1 so this is not a continuous function, unless the enrollment does not change at all.
(c) Whenever the oldest person dies the value of the function jumps down to the age of the next oldest person, so this is not a continuous function.

36. The voltage $f(t)$ is graphed in Figure 1.90.

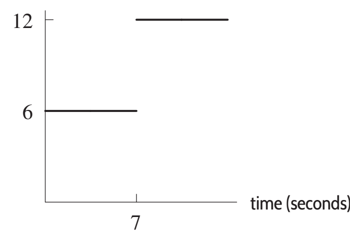


Figure 1.90: Voltage change from 6V to 12V

Using formulas, the voltage, $f(t)$, is represented by

$$f(t) = \begin{cases} 6, & 0 < t \leq 7 \\ 12, & 7 < t \end{cases}$$

Although a real physical voltage is continuous, the voltage in this circuit is well-approximated by the function $f(t)$, which is not continuous on any interval around 7 seconds.

37. (a) The function f has a jump in the graph at $t = 5$ so it is not continuous. The discontinuity at $t = 5$ occurs because the speed of the rock suddenly drops to zero when it hits the ground.
(b) The stone starts at the top of the cliff and its height decreases over time until $t = 5$ when it hits the ground. For $5 \leq t \leq 7$, we have $h = 0$ since the stone is on the ground. Therefore Figure 1.91 is a possible graph. The function g is continuous because there is no sudden change in the height of the stone.

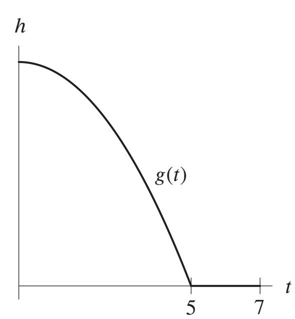


Figure 1.91

38. (a) For $0 \le t \le 2$, the car is accelerating, so f is increasing. For $2 \le t \le 3$, the speed of the car is constant, so f is constant. Finally, at $t = 3$, the speed of the car drops to zero when it hits the barrier. A possible graph of f is shown in Figure 1.92.

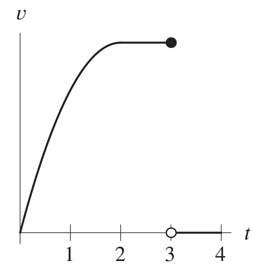


Figure 1.92

(b) No, f is not continuous because the speed of the car suddenly drops when it hits the barrier, resulting in a jump in the graph of f at $t = 3$ seconds.

39. The graph of g suggests that g is not continuous on any interval containing $\theta = 0$, since $g(0) = 1/2$.

40. For $x > 0$, we have $|x| = x$, so $f(x) = 1$. For $x < 0$, we have $|x| = -x$, so $f(x) = -1$. Thus, the function is given by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 1 \end{cases},$$

so f is not continuous on any interval containing $x = 0$.

41. The break in the graph at $x = 0$ suggests that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. See Figure 1.93.

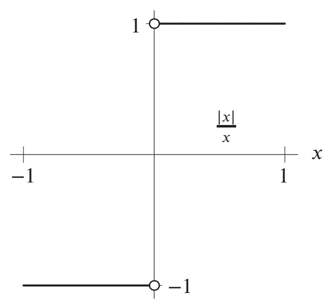


Figure 1.93

42. For $-1 \le x \le 1,\ -1 \le y \le 1$, the graph of $y = x \ln |x|$ is in Figure 1.94. The graph suggests that

$$\lim_{x\to 0} x \ln |x| = 0.$$

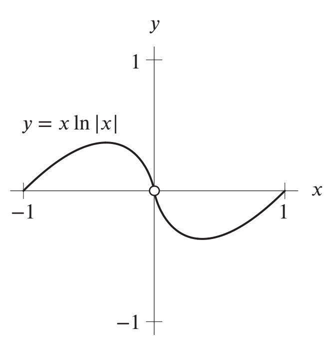


Figure 1.94

43. For $-0.5 \le \theta \le 0.5$, $0 \le y \le 3$, the graph of $y = \dfrac{\sin(2\theta)}{\theta}$ is shown in Figure 1.95. The graph suggests that

$$\lim_{\theta \to 0} \frac{\sin(2\theta)}{\theta} = 2.$$

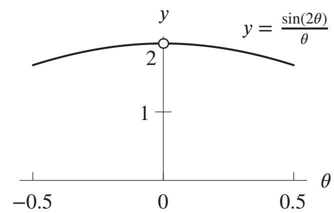


Figure 1.95

44. For $-90^\circ \le \theta \le 90^\circ$, $0 \le y \le 0.02$, the graph of $y = \dfrac{\sin\theta}{\theta}$ is shown in Figure 1.96. By tracing along the curve, we see that in degrees,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 0.01745 \ldots .$$

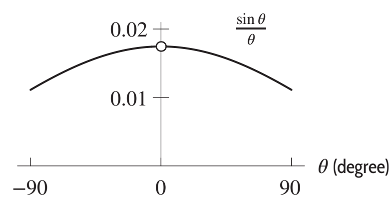


Figure 1.96

45. A graph of $y = \dfrac{e^h - 1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 3$ appears to indicate $y \to 1$ as $h \to 0$. We estimate that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

46. A graph of $y = \dfrac{e^{5h} - 1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 6$ appears to indicate $y \to 5$ as $h \to 0$. We estimate that

$$\lim_{h \to 0} \frac{e^{5h} - 1}{h} = 5.$$

47. A graph of $y = \dfrac{2^h - 1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 1$ appears to indicate $y \to 0.7$ as $h \to 0$. By zooming in on the graph, we can estimate the limit more accurately. Therefore, we estimate that

$$\lim_{h \to 0} \frac{2^h - 1}{h} = 0.693.$$

48. A graph of $y = \dfrac{\cos(3h) - 1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $-1 \le y \le 1$ appears to indicate $y \to 0$ as $h \to 0$. Therefore, we estimate that

$$\lim_{h \to 0} \frac{\cos(3h) - 1}{h} = 0.$$

49. At $x = \pi$, the curve $y = k\cos x$ has $y = k\cos\pi = -k$. At $x = \pi$, the line $y = 12 - x$ has $y = 12 - \pi$. If $h(x)$ is continuous, we need

$$\begin{aligned} -k &= 12 - \pi \\ k &= \pi - 12. \end{aligned}$$

50. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.

Since kx takes the value $k \cdot 1 = k$ at $x = 1$, we choose k so that the graph of $2kx + 3$ goes through the point $(1, k)$. This gives

$$\begin{aligned} 2k \cdot 1 + 3 &= k \\ k &= -3. \end{aligned}$$

51. For any value of k, the function is continuous at every point except $x = \pi$. We choose k to make the function continuous at $x = \pi$.

Since $k \sin x$ takes the value $k \sin \pi = 0$ at $x = \pi$, we cannot choose k so that the graph of $x + 4$ goes through the point $(\pi, 0)$. Thus, this function is discontinuous for all values of k.

52. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $x + 1$ takes the value $2 + 1 = 3$ at $x = 2$, we choose k so that the graph of e^{kx} goes through the point $(2, 3)$. This gives

$$\begin{aligned} e^{2k} &= 3 \\ 2k &= \ln 3 \\ k &= \frac{\ln 3}{2} \end{aligned}$$

53. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.

Since $\sin(kx)$ takes the value $\sin k$ at $x = 1$, we choose k so that the graph of $0.5x$ goes through the point $(1, \sin k)$. This gives

$$\begin{aligned} \sin k &= 0.5 \\ k &= \sin^{-1} 0.5 = \frac{\pi}{6}. \end{aligned}$$

Other solutions are possible.

54. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $\ln(kx + 1)$ takes the value $\ln(2k + 1)$ at $x = 2$, we choose k so that the graph of $x + 4$ goes through the point $(2, \ln(2k + 1))$. This gives

$$\begin{aligned} \ln(2k + 1) &= 2 + 4 = 6 \\ 2k + 1 &= e^6 \\ k &= \frac{e^6 - 1}{2}. \end{aligned}$$

55. (a) We see that $f(x)$ has jumps at the points $x = 1, x = 3$, and $x = 5$. Therefore, $f(x)$ is discontinuous at $x = 1, 3$, and 5.

(b) Observe that, on the intervals $0 \le x < 1$ and $3 < x \le 5$, the function $f(x)$ is positive, so $|f(x)| = f(x)$. On the interval $1 \le x \le 3$, we have $|f(x)| = |-2| = 2$, and on the interval $5 < x \le 6$, we have $|f(x)| = |-3| = 3$.

Using this information, we sketch the graph of $|f(x)|$ in Figure 1.97. We see that $|f(x)|$ is discontinuous only at $x = 3$, where the graph of $|f(x)|$ has a jump.

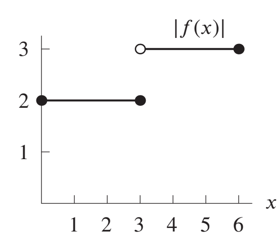


Figure 1.97

56. (a) The initial value is when $t = 0$, and we see that $P(0) = e^{k\cdot 0} = e^0 = 1000$.
(b) Since the function is continuous, at $t = 12$, we have $e^{kt} = 100$ and we solve for k:

$$\begin{aligned} e^{12k} &= 100 \\ 12k &= \ln 100 \\ k &= \frac{\ln 100}{12} = 0.384. \end{aligned}$$

(c) The population is increasing exponentially for 12 months and then becoming constant.

57. The drug first increases linearly for half a second, at the end of which time there is 0.6 ml in the body. Thus, for $0 \le t \le 0.5$, the function is linear with slope $0.6/0.5 = 1.2$:

$$Q = 1.2t \quad \text{for} \quad 0 \le t \le 0.5.$$

At $t = 0.5$, we have $Q = 0.6$. For $t > 0.5$, the quantity decays exponentially at a continuous rate of 0.002, so Q has the form

$$Q = Ae^{-0.002t} \qquad 0.5 < t.$$

We choose A so that $Q = 0.6$ when $t = 0.5$:

$$\begin{aligned} 0.6 &= Ae^{-0.002(0.5)} = Ae^{-0.001} \\ A &= 0.6e^{0.001}. \end{aligned}$$

Thus

$$Q = \begin{cases} 1.2t & 0 \le t \le 0.5 \\ 0.6e^{0.001}e^{-.002t} & 0.5 < t. \end{cases}$$

58. At $x = -1$, the function is not defined. Thus, $f(x)$ is not continuous at $x = -1$. We have

$$\lim_{x\to -1} \frac{x^2 - 1}{x + 1} = \lim_{x\to -1} \frac{(x-1)(x+1)}{x+1} = \lim_{x\to -1} (x - 1) = -2.$$

So if we let $f(-1) = -2$, the function is continuous everywhere.

59. At $x = 5$, the function is not defined. Thus, $g(x)$ is not continuous at $x = 5$. We have

$$\lim_{x\to 5} \frac{x^2 - 4x - 5}{x - 5} = \lim_{x\to 5} \frac{(x-5)(x+1)}{x-5} = \lim_{x\to 5}(x + 1) = 6.$$

So if we let $g(5) = 6$, the function is continuous everywhere.

60. At $z = 9$, the function is not defined. Thus, $f(z)$ is not continuous at $z = 9$. We have

$$\lim_{z\to 9} \frac{z^2 - 11z + 18}{2z - 18} = \lim_{z\to 9} \frac{(z-2)(z-9)}{2(z-9)} = \lim_{z\to 9} \frac{z-2}{2} = \frac{7}{2}.$$

So if we let $f(9) = 7/2$, the function is continuous everywhere.

61. At $t = 3$ and $t = -3$, the function is not defined. Thus, $q(t)$ is not continuous at $t = \pm 3$. We have

$$\lim_{t\to 3} \frac{-t^3 + 9t}{t^2 - 9} = \lim_{t\to 3} \frac{-t(t-3)(t+3)}{(t-3)(t+3)} = \lim_{t\to 3} -t = -3.$$

Similarly, we have

$$\lim_{t\to -3} \frac{-t^3 + 9t}{t^2 - 9} = \lim_{t\to -3} \frac{-t(t-3)(t+3)}{(t-3)(t+3)} = \lim_{t\to -3} -t = 3.$$

So if we let $q(3) = -3$ and $q(-3) = 3$, the function is continuous everywhere.

62. At $x = 0$, the function is not defined. In addition, $\lim_{x\to 0} f(x)$ does not exist. Thus, $f(x)$ is not continuous at $x = 0$.

63. Since $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1$, we see that $\lim_{x\to 0} f(x)$ does not exist. Thus, $f(x)$ is not continuous at $x = 0$

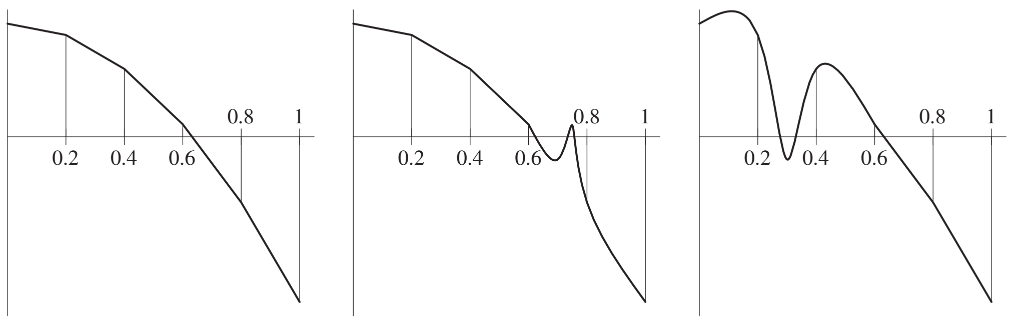
64. Since $x/x = 1$ for $x \ne 0$, this function $f(x) = 1$ for all x. Thus, $f(x)$ is continuous for all x.

65. Since $2x/x = 2$ for $x \ne 0$, we have $\lim_{x\to 0} f(x) = 2$, so

$$\lim_{x\to 0} f(x) \ne f(0) = 3.$$

Thus, $f(x)$ is not continuous at $x = 0$.

66.



67. Since polynomials are continuous, and since $p(5) < 0$ and $p(10) > 0$ and $p(12) < 0$, there are two zeros, one between $x = 5$ and $x = 10$, and another between $x = 10$ and $x = 12$. Thus, $p(x)$ is a cubic with at least two zeros.

If $p(x)$ has only two zeros, one would be a double zero (corresponding to a repeated factor). However, since a polynomial does not change sign at a repeated zero, $p(x)$ cannot have a double zero and have the signs it does.

Thus, $p(x)$ has three zeros. The third zero can be greater than 12 or less than 5. See Figures 1.98 and 1.99.

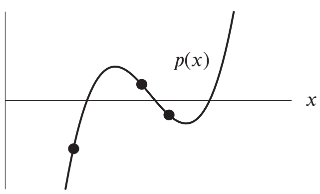


Figure 1.98

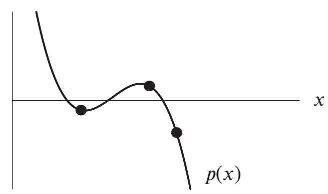


Figure 1.99

68. (a) The graphs of $y = e^x$ and $y = 4 - x^2$ cross twice in Figure 1.100. This tells us that the equation $e^x = 4 - x^2$ has two solutions.

Since $y = e^x$ increases for all x and $y = 4 - x^2$ increases for $x < 0$ and decreases for $x > 0$, these are only the two crossing points.

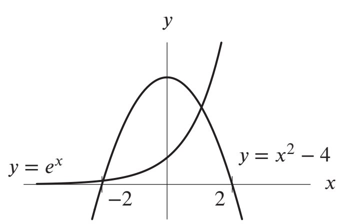


Figure 1.100

(b) Values of $f(x)$ are in Table 1.6. One solution is between $x = -2$ and $x = -1$; the second solution is between $x = 1$ and $x = 2$.

Table 1.6

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$	12.0	5.0	0.1	−2.6	−3	−0.3	7.4	25.1	66.6

69. (a) Since $f(x)$ is not continuous at $x = 1$, it does not satisfy the conditions of the Intermediate Value Theorem.

(b) We see that $f(0) = e^0 = 1$ and $f(2) = 4 + (2-1)^2 = 5$. Since e^x is increasing between $x = 0$ and $x = 1$, and since $4 + (x-1)^2$ is increasing between $x = 1$ and $x = 2$, any value of k between $e^1 = e$ and $4 + (1-1)^2 = 4$, such as $k = 3$, is a value such that $f(x) = k$ has no solution.

70. A function $y = g(x)$ is invertible only if each y-value corresponds to a unique x-value. (This condition is called the "horizontal line test" as it tells us that every horizontal line cuts the graph of an invertible function at most once.) We will find a value y corresponding to two different x-values.

Since $g(x)$ is continuous with $g(0) = 3$ and $g(1) = 8$, the Intermediate Value Theorem guarantees that between $x = 0$ and $x = 1$, the function $g(x)$ takes every value between 3 and 8. We pick any one of the y-values between 3 and 8 that is also between 8 and 4. We will pick $y = 6$, but it could equally well be 5.5 or 7.

By the Intermediate Value Theorem, there exists a number c between $x = 0$ and $x = 1$ with $g(c) = 6$.

Likewise, since $g(x)$ is continuous with $g(1) = 8$ and $g(2) = 4$, the Intermediate Value Theorem guarantees that between $x = 1$ and $x = 2$, the function $g(x)$ takes every value between 4 and 8. In particular, there is a number d between $x = 1$ and $x = 2$ with $g(d) = 6$.

This means there are two x-values, $x = c$ and $x = d$, with c between 0 and 1 and d between 1 and 2, so $c \neq d$ and $g(c) = g(d)$. Thus $g(x)$ is not invertible.

71. The answer (see the graph in Figure 1.101) appears to be about 2.7; if we zoom in further, it appears to be about 2.72, which is close to the value of $e \approx 2.71828$.

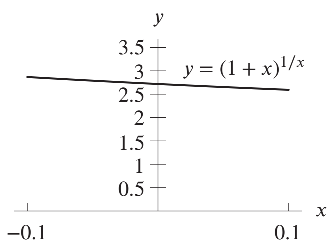


Figure 1.101

72. We use values of h approaching, but not equal to, zero. If we let $h = 0.01, 0.001, 0.0001, 0.00001$, we calculate the values 2.7048, 2.7169, 2.7181, and 2.7183. If we let $h = -0.01$ -0.001, -0.0001, -0.00001, we get values 2.7320, 2.7196, 2.7184, and 2.7183. These numbers suggest that the limit is the number $e = 2.71828\ldots$. However, these calculations cannot tell us that the limit is exactly e; for that a proof is needed.

73. (a) Since $\sin(n\pi) = 0$ for $n = 1, 2, 3, \ldots$ the sequence of x-values

$$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$$

works. These x-values $\to 0$ and are zeroes of $f(x)$.

(b) Since $\sin(n\pi/2) = 1$ for $n = 1, 5, 9 \ldots$ the sequence of x-values

$$\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$$

works.

(c) Since $\sin(n\pi)/2 = -1$ for $n = 3, 7, 11, \ldots$ the sequence of x-values

$$\frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi} \ldots$$

works.

(d) Any two of these sequences of x-values show that if the limit were to exist, then it would have to have two (different) values: 0 and 1, or 0 and -1, or 1 and -1. Hence, the limit can not exist.

Strengthen Your Understanding

74. The Intermediate Value theorem only makes this guarantee for a *continuous* function, not for any function.

75. The Intermediate Value Theorem guarantees that for at least one value of x between 0 and 2, we have $f(x) = 5$, but it does not tell us which value(s) of x give $f(x) = 5$.

76. For f to be continuous at $x = c$, we need $\lim_{x\to c} f(x)$ to exist and to be equal to $f(c)$.

77. We want a function which has a value at every point but where the graph has a break at $x = 15$. One possibility is

$$f(x) = \begin{cases} 1 & x \geq 15 \\ -1 & x < 15 \end{cases}$$

78. One example is $f(x) = 1/x$, which is not continuous at $x = 0$. The Intermediate Value Theorem does not apply on an interval that contains a point where a function is not continuous.

79. Let $f(x) = \begin{cases} 1 & x \leq 2 \\ x & x > 2 \end{cases}$. Then $f(x)$ is continuous at every point in $[0, 3]$ except at $x = 2$. Other answers are possible.

80. Let $f(x) = \begin{cases} x & x \leq 3 \\ 2x & x > 3 \end{cases}$. Then $f(x)$ is increasing for all x but $f(x)$ is not continuous at $x = 3$. Other answers are possible.

81. False. For example, let $f(x) = \begin{cases} 1 & x \leq 3 \\ 2 & x > 3 \end{cases}$, then $f(x)$ is defined at $x = 3$ but it is not continuous at $x = 3$. (Other examples are possible.)

82. False. A counterexample is graphed in Figure 1.102, in which $f(5) < 0$.

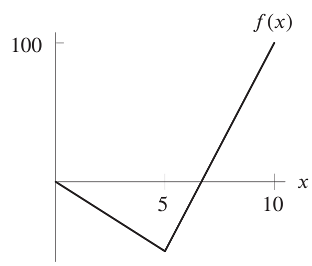


Figure 1.102

83. False. A counterexample is graphed in Figure 1.103.

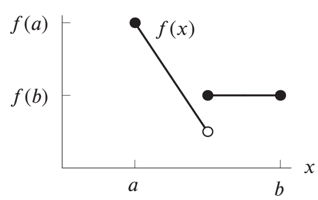


Figure 1.103

Solutions for Section 1.8

Exercises

1. (a) As x approaches -1 from the right, $f(x)$ approaches 1. So $\lim_{x \to -1^+} f(x) = 1$.

(b) As x approaches 0 from the left, $f(x)$ approaches 0. So $\lim_{x \to 0^-} f(x) = 0$.

(c) As x approaches 0 from either side, $f(x)$ approaches 0, regardless of the fact that $f(0) = 2$. So $\lim_{x \to 0} f(x) = 0$.

(d) As x approaches 1 from the left, $f(x)$ approaches 1. So $\lim_{x \to 1^-} f(x) = 1$.

(e) As x approaches 1 from the left, $f(x)$ approaches 1. However, as x approaches 1 from the right, $f(x)$ approaches 2. Since there is no single value that $f(x)$ approaches as x approaches 1 from both sides, $\lim_{x \to 1} f(x)$ does not exist.

(f) As x approaches 2 from the left, $f(x)$ approaches 1. So $\lim_{x \to 2^-} f(x) = 1$.

2. (a) As x approaches 1 from the left, the values of $f(x)$ get closer and closer to 1. So $\lim_{x\to 1^-} f(x) = 1$.
(b) As x approaches 1 from the right, $f(x)$ goes to $-\infty$. So limit does not exist since it does not approach a real number L, but we can use limit notation to say $\lim_{x\to 1^+} f(x) = -\infty$.
(c) As x approaches 1 from the left, the values of $f(x)$ get closer and closer to 1. However, as x approaches 1 from the right, $f(x)$ goes to $-\infty$. Since there is no single value that $f(x)$ approaches as x approaches 1 from both sides, $\lim_{x\to 1} f(x)$ does not exist.
(d) As x approaches 2 from the left, the values of $f(x)$ get closer and closer to 1. So $\lim_{x\to 2^-} f(x) = 1$.
(e) As x approaches 2 from the right, the values of $f(x)$ get closer and closer to 1. So $\lim_{x\to 2^+} f(x) = 1$.
(f) As x approaches 2 from either side, $f(x)$ approaches 1, regardless of the fact that $f(2)$ is undefined. So $\lim_{x\to 2} f(x) = 1$.

3. (a) When $x = -2$, we see that $y = 4$. Therefore, $f(-2) = 4$.
(b) When $x = 0$, we see that $y = 2$. Therefore, $f(0) = 2$.
(c) As x approaches -4 from the right, $f(x)$ approaches -4. Therefore, $\lim_{x\to -4^+} f(x) = -4$.
(d) As x approaches -2 from the left, $f(x)$ approaches 2. Therefore, $\lim_{x\to -2^-} f(x) = 2$.
(e) As x approaches -2 from the right, $f(x)$ approaches 4. Therefore, $\lim_{x\to -2^+} f(x) = 4$.
(f) As x approaches 0 from both sides, $f(x)$ approaches 0. Therefore, $\lim_{x\to 0} f(x) = 0$.
(g) As x approaches 2 from the left, $f(x)$ approaches 2. However, as x approaches 2 from the right, $f(x)$ approaches 4. Therefore, $\lim_{x\to 2} f(x)$ does not exist.
(h) As x approaches 4 from the left, $f(x)$ approaches 2. Therefore, $\lim_{x\to 4^-} f(x) = 2$.

4. (a) When $x = 0$, there is no corresponding value of y on the graph of f. Therefore, $f(0)$ is undefined.
(b) When $x = 4$, we see that $y = -2$. Therefore, $f(4) = -2$.
(c) As x approaches -2 from the left, $f(x)$ approaches 2. Therefore, $\lim_{x\to -2^-} f(x) = 2$.
(d) As x approaches -2 from the right, $f(x)$ approaches -2. Therefore, $\lim_{x\to -2^+} f(x) = -2$.
(e) From parts (c) and (d), we see that $\lim_{x\to -2^-} f(x)$ is not equal to $\lim_{x\to -2^+} f(x)$. Therefore, $\lim_{x\to -2} f(x)$ does not exist.
(f) As x approaches 0 from both sides, $f(x)$ approaches 2. Therefore, $\lim_{x\to 0} f(x) = 2$.
(g) As x approaches 2 from the left, $f(x)$ approaches $-\infty$. Therefore, $\lim_{x\to 2^-} f(x) = -\infty$.
(h) As x approaches 2 from the right, $f(x)$ approaches ∞. Therefore, $\lim_{x\to 2^+} f(x) = \infty$.
(i) From parts (g) and (h), we see that $\lim_{x\to 2^-} f(x)$ is not equal to $\lim_{x\to 2^+} f(x)$. Therefore, $\lim_{x\to 2} f(x)$ does not exist.
(j) As x approaches 4 from the left, $f(x)$ approaches 2. However, as x approaches 4 from the right, $f(x)$ approaches -2. Therefore, $\lim_{x\to 4} f(x)$ does not exist.

5. (a) As x gets larger and larger, the values of $f(x)$ oscillate, but the oscillations get smaller and smaller around 3. Therefore, we estimate that $\lim_{x\to\infty} f(x) = 3$
(b) As x gets larger and larger in magnitude but negative, the values of $f(x)$ get larger and larger in magnitude but negative, so the $f(x)$ does not approach a real number, so the limit does not exist. Alternatively, we can say $\lim_{x\to -\infty} f(x) = -\infty$.

6. Using Theorem 1.2 we have,

$$\begin{aligned}
\lim_{x\to 2}(f(x) - 2h(x)) &= \lim_{x\to 2} f(x) - \lim_{x\to 2} 2h(x) && \text{Property 2}\\
&= \lim_{x\to 2} f(x) - 2\lim_{x\to 2} h(x)) && \text{Property 1}\\
&= 7 - 2\cdot\frac{1}{2} = 6.
\end{aligned}$$

7. Using Theorem 1.2 we have,

$$\begin{aligned}
\lim_{x\to 2}(g(x))^2 &= \lim_{x\to 2}(g(x)\cdot g(x))\\
&= (\lim_{x\to 2} g(x))\cdot(\lim_{x\to 2} g(x)) && \text{Property 3}\\
&= (-4)(-4) = 16.
\end{aligned}$$

8. Using Theorem 1.2 we have,

$$\lim_{x\to 2}\frac{f(x)}{g(x)\cdot h(x)} = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2}(g(x)\cdot h(x))} \qquad \text{Property 4}$$

$$= \frac{\lim_{x\to 2} f(x)}{(\lim_{x\to 2} g(x))(\lim_{x\to 2} h(x))} \quad \text{Property 3}$$
$$= \frac{7}{(-4)(\frac{1}{2})} = -\frac{7}{2}.$$

9. From the graphs of f and g, we estimate $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^-} g(x) = 5$, $\lim_{x\to 1^+} f(x) = 4$, $\lim_{x\to 1^+} g(x) = 1$.

(a) Using Theorem 1.2 we have $\lim_{x\to 1^-} (f(x) + g(x)) = 3 + 5 = 8$.

(b) Using Theorem 1.2 we have $\lim_{x\to 1^+} (f(x) + 2g(x)) = \lim_{x\to 1^+} f(x) + 2 \lim_{x\to 1^+} g(x) = 4 + 2(1) = 6$

(c) Using Theorem 1.2 we have $\lim_{x\to 1^-} (f(x)g(x)) = (\lim_{x\to 1^-} f(x))(\lim_{x\to 1^-} g(x)) = (3)(5) = 15$

(d) Using Theorem 1.2 we have $\lim_{x\to 1^+} (f(x)/g(x)) = \left(\lim_{x\to 1^+} f(x)\right) / \left(\lim_{x\to 1^+} g(x)\right) = 4/1 = 4$

10. We see that $f(x)$ goes to $-\infty$ on both ends, so one possible graph is shown in Figure 1.104. Other answers are possible.

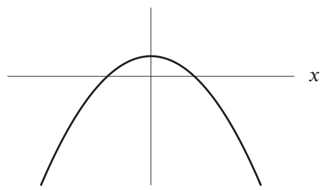


Figure 1.104

11. We see that $f(x)$ goes to ∞ on the left and to $-\infty$ on the right. One possible graph is shown in Figure 1.105. Other answers are possible.

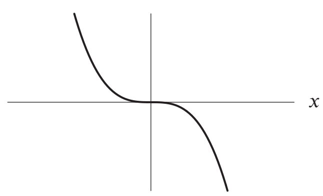


Figure 1.105

12. We see that $f(x)$ goes to ∞ on the left and approaches a y-value of 1 on the right. One possible graph is shown in Figure 1.106. Other answers are possible.

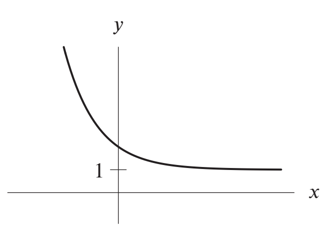


Figure 1.106

13. We see that $f(x)$ approaches a y-value of 3 on the left and goes to $-\infty$ on the right. One possible graph is shown in Figure 1.107. Other answers are possible.

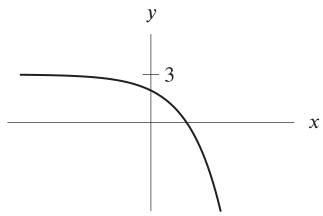


Figure 1.107

14. We see that $f(x)$ goes to ∞ on the right and that it also passes through the point $(-1, 2)$. (Notice that this must be a point on the graph since the instructions require that $f(x)$ be defined and continuous.) One possible graph is shown in Figure 1.108. Other answers are possible.

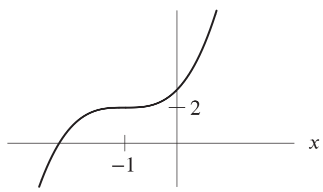


Figure 1.108

15. We see that $f(x)$ goes to ∞ on the left and that it also passes through the point $(3, 5)$. (Notice that this must be a point on the graph since the instructions require that $f(x)$ be defined and continuous.) One possible graph is shown in Figure 1.109. Other answers are possible.

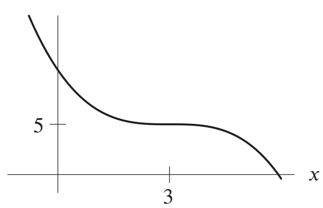


Figure 1.109

16. We have $\lim_{x\to\infty} x^2 = \infty$ because x^2 increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive.

17. We have $\lim_{x\to-\infty} x^2 = \infty$ because x^2 increases and becomes arbitrarily large and positive as x becomes arbitrarily large in magnitude and negative.

18. We have $\lim_{x\to-\infty} x^3 = -\infty$ because x^3 decreases and becomes arbitrarily large in magnitude and negative as x becomes arbitrarily large in magnitude and negative.

19. We have $\lim_{x\to\infty} x^3 = \infty$ because x^3 increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive.

20. We have $\lim_{x\to\infty} e^x = \infty$ because e^x increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive.

21. We have $\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \dfrac{1}{e^x} = 0$ because e^x increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive, so $1/e^x$ gets closer and closer to 0 as x becomes arbitrarily large and positive.

22. We have $\lim_{x\to\infty} 5^{-x} = \lim_{x\to\infty} \dfrac{1}{5^x} = 0$ because 5^x increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive, so $1/5^x$ gets closer and closer to 0 as x becomes arbitrarily large and positive.

23. We have $\lim_{x\to\infty} \sqrt{x} = \infty$ because $\sqrt{x}$ increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive.

24. We have $\lim_{x\to\infty} \ln x = \infty$ because $\ln x$ increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive.

25. We have $\lim_{x\to\infty} x^{-2} = \lim_{x\to\infty} \frac{1}{x^2} = 0$ because x^2 increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive, so $1/x^2$ gets closer and closer to 0 as x becomes arbitrarily large and positive.

26. We have $\lim_{x\to-\infty} x^{-2} = \lim_{x\to-\infty} \frac{1}{x^2} = 0$ because x^2 increases and becomes arbitrarily large and positive as x becomes arbitrarily large in magnitude and negative, so $1/x^2$ gets closer and closer to 0 as x becomes arbitrarily large in magnitude and negative.

27. We have $\lim_{x\to-\infty} x^{-3} = \lim_{x\to-\infty} \frac{1}{x^3} = 0$ because x^3 decreases and becomes arbitrarily large in magnitude and negative as x becomes arbitrarily large in magnitude and negative, so $1/x^3$ gets closer and closer to 0 as x becomes arbitrarily large in magnitude and negative.

28. We have $\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = \lim_{x\to\infty} \frac{1}{2^x} = 0$ because 2^x increases and becomes arbitrarily large and positive as x becomes arbitrarily large and positive, so $1/2^x$ gets closer and closer to 0 as x becomes arbitrarily large and positive.

29. We see that $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = -\infty$.

30. As $x \to \pm\infty$, we know that $f(x)$ behaves like its leading term $-2x^3$. Thus, we have $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$.

31. As $x \to \pm\infty$, we know that $f(x)$ behaves like its leading term x^5. Thus, we have $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$.

32. As $x \to \pm\infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{3x^3}{5x^3} = \frac{3}{5},$$

we have $\lim_{x\to-\infty} f(x) = 3/5$ and $\lim_{x\to\infty} f(x) = 3/5$.

33. As $x \to \pm\infty$, we know that x^{-3} gets closer and closer to zero, so we have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$.

34. We have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$.

35. Since $\lim_{x\to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x\to 0^-} \frac{|x|}{x} = -1$, we say that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. In addition $f(x)$ is not defined at 0. Therefore, $f(x)$ is not continuous on any interval containing 0.

36. $$f(x) = \frac{|x-4|}{x-4} = \begin{cases} \frac{x-4}{x-4} & x > 4 \\ -\frac{x-4}{x-4} & x < 4 \end{cases} = \begin{cases} 1 & x > 4 \\ -1 & x < 4 \end{cases}$$

Figure 1.110 confirms that $\lim_{x\to 4^+} f(x) = 1$, $\lim_{x\to 4^-} f(x) = -1$ so $\lim_{x\to 4} f(x)$ does not exist.

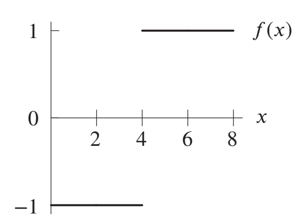


Figure 1.110

37. $$f(x) = \frac{|x-2|}{x} = \begin{cases} \frac{x-2}{x}, & x > 2 \\ -\frac{x-2}{x}, & x < 2 \end{cases}$$

Figure 1.111 confirms that $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x) = 0$.

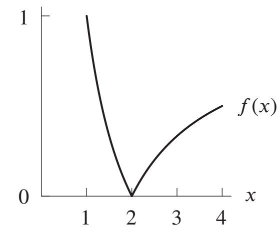


Figure 1.111

38. $f(x) = \begin{cases} x^2 - 2 & 0 < x < 3 \\ 2 & x = 3 \\ 2x + 1 & 3 < x \end{cases}$

Figure 1.112 confirms that $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 - 2) = 7$ and that $\lim_{x\to 3+} f(x) = \lim_{x\to 3+} (2x + 1) = 7$, so $\lim_{x\to 3} f(x) = 7$. Note, however, that $f(x)$ is not continuous at $x = 3$ since $f(3) = 2$.

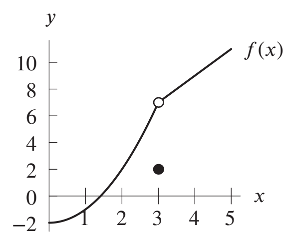


Figure 1.112

Problems

39. The graph in Figure 1.113 suggests that as x grows large, the value of $(1 + 1/x)^x$ approaches a limit, which appears to be slightly greater than 2.7. If we zoom out further, we see it appears to be about 2.72, which is close to the value of $e \approx 2.71828$. Thus we guess that $\lim_{x\to\infty} (1 + 1/x)^x = e$

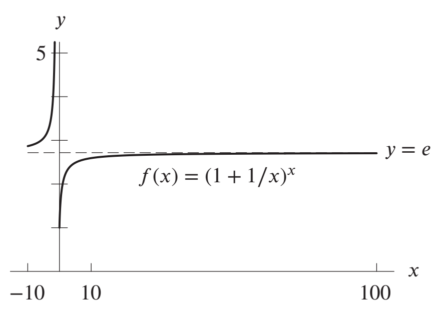


Figure 1.113

40. We use values of x approaching ∞ and make a table. We see that as x grows very large, the value of $(1 + 1/x)^x$ appears to approach a value close to 2.7183. We recognize this value as e rounded to four decimal places, so we estimate $\lim_{x\to\infty} (1 + 1/x)^x = e$.

x	1	10	100	1000	10000	100000	1000000
$(1 + 1/x)^x$	2.0000	2.5937	2.7048	2.7169	2.7181	2.7183	2.7183

41. (a)

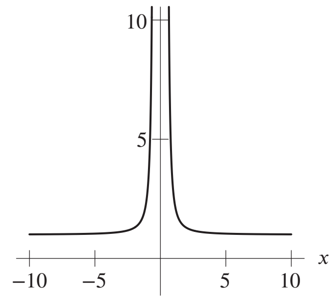


Figure 1.114

(b) The graph of $f(x)$ suggests that it is not continuous at $x = 0$. However, the function $f(x)$ is the composition of a rational function $1/(x^2 + 0.0001)$, which is continuous everywhere since its denominator is never zero, with the exponential function e^x, which is continuous everywhere. Therefore $f(x)$ is continuous everywhere, so must be continuous at $x = 0$.

42. When $x = 0.1$, we find $xe^{1/x} \approx 2203$. When $x = 0.01$, we find $xe^{1/x} \approx 3 \times 10^{41}$. When $x = 0.001$, the value of $xe^{1/x}$ is too big for a calculator to compute. This suggests that $\lim_{x\to 0^+} xe^{1/x}$ does not exist (and in fact it does not).

43. As $x \to \infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{x}{-x} = -1,$$

we have $\lim_{x\to\infty} f(x) = -1$.

Alternatively, we could divide numerator and denominator by x and apply the properties of limits:

$$f(x) = \frac{x+3}{2-x} = \frac{1+3/x}{2/x-1},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+3/x}{2/x-1} = \frac{\lim_{x\to\infty}(1+3/x)}{\lim_{x\to\infty}(2/x-1)} = \frac{\lim_{x\to\infty} 1 + \lim_{x\to\infty}(3/x)}{\lim_{x\to\infty}(2/x) - \lim_{x\to\infty} 1} = \frac{1}{-1} = -1$$

since $1/x \to 0$ as $x \to \infty$.

44. As $x \to \infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{3x}{\pi x} = \frac{3}{\pi},$$

we have $\lim_{x\to\infty} f(x) = 3/\pi$.

Alternatively, we could divide numerator and denominator by x and apply the properties of limits:

$$f(x) = \frac{\pi+3x}{\pi x-3} = \frac{\pi/x+3}{\pi-3/x},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\pi/x+3}{\pi-3/x} = \frac{\lim_{x\to\infty}(\pi/x+3)}{\lim_{x\to\infty}(\pi-3/x)} = \frac{\lim_{x\to\infty}(\pi/x) + \lim_{x\to\infty} 3}{\lim_{x\to\infty}\pi - \lim_{x\to\infty}(3/x)} = \frac{3}{\pi}$$

since $a/x \to 0$ as $x \to \infty$ for any nonzero number a..

45. As $x \to \infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{x}{2x^2} = \frac{1}{2x},$$

we have $\lim_{x\to\infty} f(x) = 0$ because $1/(2x) \to 0$ as $x \to \infty$.

Alternatively, we could divide numerator and denominator by x^2 and apply the properties of limits:

$$f(x) = \frac{x-5}{5+2x^2} = \frac{1/x-5/x^2}{5/x^2+2},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1/x - 5/x^2}{5/x^2 + 2} = \frac{\lim_{x\to\infty}\left((1/x) - (5/x^2)\right)}{\lim_{x\to\infty}\left((5/x^2) + 2\right)} = \frac{\lim_{x\to\infty}(1/x) - \lim_{x\to\infty}(5/x^2)}{\lim_{x\to\infty}(5/x^2) + \lim_{x\to\infty} 2} = \frac{0}{2} = 0$$

since $\lim_{x\to\infty} a/x^n = 0$ for any nonzero number a and positive n.

46. As $x \to \infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{x^2}{3x^2} = \frac{1}{3},$$

we have $\lim_{x\to\infty} f(x) = 1/3$.

Alternatively, we could divide numerator and denominator by x^2 and apply the properties of limits:

$$f(x) = \frac{x^2 + 2x - 1}{3 + 3x^2} = \frac{1 + 2/x - 1/x^2}{3/x^2 + 3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1 + 2/x - 1/x^2}{3/x^2 + 3} = \frac{\lim_{x\to\infty}\left(1 + 2/x - 1/x^2\right)}{\lim_{x\to\infty}\left(3/x^2 + 3\right)} = \frac{\lim_{x\to\infty} 1 + \lim_{x\to\infty}(2/x) - \lim_{x\to\infty}(1/x^2)}{\lim_{x\to\infty}(3/x^2) + \lim_{x\to\infty} 3} = \frac{1}{3}$$

since $\lim_{x\to\infty} a/x^n = 0$ for any nonzero number a and positive n.

47. As $x \to \infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{x^2}{x} = x,$$

we have $\lim_{x\to\infty} f(x) = \infty$.

Alternatively, we could divide numerator and denominator by x and apply the properties of limits:

$$f(x) = \frac{x^2 + 4}{x + 3} = \frac{x + 4/x}{1 + 3/x},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x + 4/x}{1 + 3/x} = \frac{\lim_{x\to\infty}(x + 4/x)}{\lim_{x\to\infty}(1 + 3/x)} = \frac{\lim_{x\to\infty} x + \lim_{x\to\infty}(4/x)}{\lim_{x\to\infty} 1 + \lim_{x\to\infty}(3/x)} = \lim_{x\to\infty} x = \infty$$

since $\lim_{x\to\infty} a/x = 0$ for any nonzero number a.

48. As $x \to \infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{2x^3}{3x^3} = \frac{2}{3},$$

we have $\lim_{x\to\infty} f(x) = 2/3$.

Alternatively, we could divide numerator and denominator by x^3 and apply the properties of limits:

$$f(x) = \frac{2x^3 - 16x^2}{4x^2 + 3x^3} = \frac{2 - 16/x}{4/x + 3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2 - 16/x}{4/x + 3} = \frac{\lim_{x\to\infty}(2 - 16/x)}{\lim_{x\to\infty}(4/x + 3)} = \frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty}(16/x)}{\lim_{x\to\infty}(4/x) + \lim_{x\to\infty} 3} = \frac{2}{3}$$

since $\lim_{x\to\infty} a/x = 0$ for any nonzero number a.

49. As $x \to \infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{x^4}{2x^5} = \frac{1}{2x},$$

we have $\lim_{x\to\infty} f(x) = 0$ because $1/(2x) \to 0$ as $x \to \infty$.

Alternatively, we could divide numerator and denominator by x^5 and apply the properties of limits:

$$f(x) = \frac{x^4 + 3x}{x^4 + 2x^5} = \frac{1/x + 3/x^4}{1/x + 2},$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}\left(1/x + 3/x^4\right)}{\lim_{x\to\infty}(1/x + 2)} = \frac{\lim_{x\to\infty}(1/x) + \lim_{x\to\infty}(3/x^4)}{\lim_{x\to\infty}(1/x) + \lim_{x\to\infty} 2} = \frac{0}{2} = 0$$

since $\lim_{x\to\infty} a/x^n = 0$ for any nonzero number a and positive n.

50. For large values of x, $3e^x + 2$ behaves like $3e^x$ and $2e^x + 3$ behaves like $2e^x$, so for large values of x,

$$f(x) \to \frac{3e^x}{2e^x} = \frac{3}{2}.$$

Thus we have we have $\lim_{x\to\infty} f(x) = 3/2$.

Alternatively, we could divide numerator and denominator by e^x and apply the properties of limits:

$$f(x) = \frac{3e^x + 2}{2e^x + 3} = \frac{3 + 2e^{-x}}{2 + 3e^{-x}},$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(3 + 2e^{-x})}{\lim_{x\to\infty}(2 + 3e^{-x})} = \frac{\lim_{x\to\infty} 3 + \lim_{x\to\infty} 2e^{-x}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty} 3e^{-x}} = \frac{3}{2}$$

since $\lim_{x\to\infty} ae^{-x} = 0$ for any nonzero number a.

51. Since $\lim_{x\to\infty} 2^{-x} = \lim_{x\to\infty} 3^{-x} = 0$, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2^{-x} + 5}{3^{-x} + 7} = \frac{\lim_{x\to\infty}(2^{-x} + 5)}{\lim_{x\to\infty}(3^{-x} + 7)} = \frac{\lim_{x\to\infty} 2^{-x} + \lim_{x\to\infty} 5}{\lim_{x\to\infty} 3^{-x} + \lim_{x\to\infty} 7} = \frac{5}{7}.$$

52. Since $\lim_{x\to\infty} ae^{-x} = 0$ for any nonzero a, we have

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(2e^{-x} + 3)}{\lim_{x\to\infty}(3e^{-x} + 2)} = \frac{\lim_{x\to\infty} 2e^{-x} + \lim_{x\to\infty} 3}{\lim_{x\to\infty} 3e^{-x} + \lim_{x\to\infty} 2} = \frac{3}{2}.$$

53. (a) Figure 1.115 shows a possible graph of $f(x)$, yours may be different.

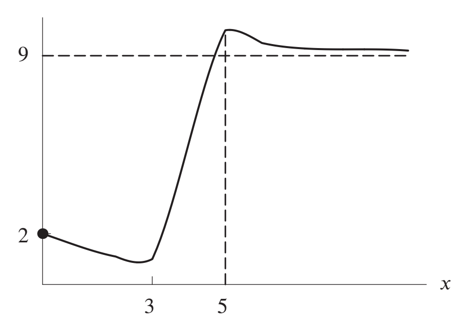


Figure 1.115

(b) In order for f to approach the horizontal asymptote at 9 from above it is necessary that f eventually become concave up. It is therefore not possible for f to be concave down for all $x > 6$.

54. One possible graph is shown in Figure 1.116. Other answers are possible.

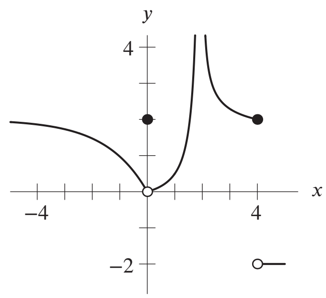


Figure 1.116

55. One possible graph is shown in Figure 1.117. Other answers are possible.

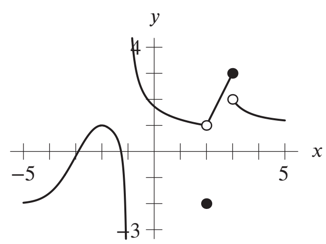


Figure 1.117

56. (a) We know that $\lim_{x\to a} f(x)$ does not exist for $a = 3$. Since the graph of $f(x)$ has a vertical asymptote at $x = 3$, this means that $f(x)$ approaches ∞ or $-\infty$ as x approaches 3 from the right or from the left. This means that $f(x)$ cannot approach a finite real number as x approaches 3 from both sides.

(b) If $\lim_{x\to\infty} f(x)$ exists, then the graph of $f(x)$ must approach a horizontal asymptote as x approaches ∞. Since the only horizontal asymptote is $y = -4$, the value of $f(x)$ must approach -4 as x gets large. So $\lim_{x\to\infty} f(x) = -4$.

57. (a) (i) As t approaches 1 from the left, Q approaches 50. Therefore, $\lim_{t\to 1^-} f(t) = 50$. This means that the amount of the drug in the patient's body is about 50 mg right before the second dose is taken.

(ii) As t approaches 1 from the right, Q approaches 150. Therefore, $\lim_{t\to 1^+} f(t) = 150$. This means that the amount of drug in the patient's body is about 150 mg right after the second dose is taken.

(b) The function f is discontinuous at $t = 1, 2, 3$, and 4 days. These points of discontinuity occur at the times when the drug is taken because the amount of the drug in the body suddenly increases at these times.

58. For all x sufficiently close to $x = 1$ but with $x > 1$, we have $p(x) = 14$, so $\lim_{x\to 1^+} p(x) = 14$. For all x sufficiently close to $x = 1$ but with $x < 1$ we have $p(x) = 7$, so $\lim_{x\to 1^-} p(x) = 7$. Therefore, the one sided limits of $p(x)$ at $x = 1$ are different and so $\lim_{x\to 1} p(x)$ does not exist.

59. We calculate this limit in stages, using the limit properties in Theorem 1.2 to justify each step:

$$\lim_{x\to 3}\frac{x^2+5x}{x+9} = \frac{\lim_{x\to 3}\left(x^2+5x\right)}{\lim_{x\to 3}(x+9)} \qquad \text{(Property 4, since } \lim_{x\to 3}(x+9)\neq 0\text{)}$$

$$= \frac{\lim_{x\to 3}\left(x^2\right)+\lim_{x\to 3}(5x)}{\lim_{x\to 3}x+\lim_{x\to 3}9} \qquad \text{(Property 2)}$$

$$= \frac{\left(\lim_{x\to 3}x\right)^2+5\left(\lim_{x\to 3}x\right)}{\lim_{x\to 3}x+\lim_{x\to 3}9} \qquad \text{(Properties 1 and 3)}$$

$$= \frac{3^2+5\cdot 3}{3+9} = 2. \qquad \text{(Properties 5 and 6)}$$

60. There are many possible correct answers.

(a) If $f(x) = x^2$ and $g(x) = x$ then $f(x)/g(x) = x$ so $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$.

(b) If $f(x) = 3x$ and $g(x) = x$ then $f(x)/g(x) = 3$ so $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 3$.

(c) If $f(x) = x$ and $g(x) = x^2$ then $f(x)/g(x) = 1/x$ so $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$.

61. (a) We have

$$\begin{aligned}\frac{1}{x-5} - \frac{10}{x^2-5} &= \frac{x+5}{(x+5)(x-5)} - \frac{10}{(x+5)(x-5)}\\ &= \frac{x+5-10}{(x+5)(x-5)}\\ &= \frac{x-5}{(x+5)(x-5)}.\end{aligned}$$

(b) We have

$$\begin{aligned}\lim_{x\to5}\left(\frac{1}{x-5} - \frac{10}{x^2-25}\right) &= \lim_{x\to5} \frac{x-5}{(x+5)(x-5)}\\ &= \lim_{x\to5} \frac{1}{x+5} && \text{Canceling } x-5 \text{ since } x \neq 5 \text{ in the limit}\\ &= \frac{1}{5+5} = \frac{1}{10} && \text{Substituting } x=5 \text{ since } 1/(x+5) \text{ is continuous at } x=5.\end{aligned}$$

(c) Property 2 of Theorem 1.2 states

$$\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

provided the limits on the right hand side exist. In this case, neither $\lim_{x\to5} \dfrac{1}{x-5}$ nor $\lim_{x\to5} -\dfrac{10}{x^2-25}$ exist, so we cannot invoke the property.

62. The only change is that, instead of considering all x near c, we only consider x near to and greater than c. Thus we define $\lim_{x\to c^+} f(x)$, to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c and $x > c$.

63. The only change is that, instead of considering all x near c, we only consider x near to and less than c. Thus we define $\lim_{x\to c^-} f(x)$, to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c and $x < c$.

64. If c is in the interval, we know $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Then,

$$\begin{aligned}\lim_{x\to c}(f(x)+g(x)) &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) && \text{Property 2}\\ &= f(c) + g(c), && \text{so } f+g \text{ is continuous at } x=c.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{x\to c}(f(x)g(x)) &= \lim_{x\to c} f(x) \lim_{x\to c} g(x) && \text{Property 3}\\ &= f(c)g(c) && \text{so } fg \text{ is continuous at } x=c.\end{aligned}$$

Finally,

$$\begin{aligned}\lim_{x\to c} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} && \text{Property 4}\\ &= \frac{f(c)}{g(c)}, && \text{so } \frac{f}{g} \text{ is continuous at } x=c.\end{aligned}$$

Strengthen Your Understanding

65. Though $P(x)$ and $Q(x)$ are both continuous for all x, it is possible for $Q(x)$ to be equal to zero for some x. For any such value of x, where $Q(x) = 0$, we see that $P(x)/Q(x)$ is undefined, and thus not continuous. For example,

$$\frac{P(x)}{Q(x)} = \frac{x}{x-1}$$

is not defined or continuous at $x = 1$.

66. The left- and right-hand limits are not the same:

$$\lim_{x\to 1^-} \frac{x-1}{|x-1|} = -1,$$

but

$$\lim_{x\to 1^+} \frac{x-1}{|x-1|} = 1.$$

Since the left- and right-hand limits are not the same, the limit does not exist and thus is not equal to 1.

67. One possibility is

$$f(x) = \frac{(x+3)(x-1)}{x-1}.$$

We have $\lim_{x\to 1} f(x) = 4$ but $f(1)$ is undefined.

68. One example is

$$f(x) = \frac{2|x|}{x}.$$

69. True, by Property 3 of limits in Theorem 1.2, since $\lim\limits_{x\to 3} x = 3$.

70. False. If $\lim\limits_{x\to 3} g(x)$ does not exist, then $\lim\limits_{x\to 3} f(x)g(x)$ may not even exist. For example, let $f(x) = 2x + 1$ and define g by:

$$g(x) = \begin{cases} 1/(x-3) & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$$

Then $\lim\limits_{x\to 3} f(x) = 7$ and $g(3) = 4$, but $\lim\limits_{x\to 3} f(x)g(x) \neq 28$, since $\lim\limits_{x\to 3}(2x+1)/(x-3)$ does not exist.

71. True, by Property 2 of limits in Theorem 1.2.

72. True, by Properties 2 and 3 of limits in Theorem 1.2.

$$\lim_{x\to 3} g(x) = \lim_{x\to 3}(f(x) + g(x) + (-1)f(x)) = \lim_{x\to 3}(f(x) + g(x)) + (-1)\lim_{x\to 3} f(x) = 12 + (-1)7 = 5.$$

73. True. Suppose instead that $\lim\limits_{x\to 3} g(x)$ does not exist but $\lim\limits_{x\to 3}(f(x)g(x))$ did exist. Since $\lim\limits_{x\to 3} f(x)$ exists and is not zero, then $\lim\limits_{x\to 3}((f(x)g(x))/f(x))$ exists, by Property 4 of limits in Theorem 1.2. Furthermore, $f(x) \neq 0$ for all x in some interval about 3, so $(f(x)g(x))/f(x) = g(x)$ for all x in that interval. Thus $\lim\limits_{x\to 3} g(x)$ exists. This contradicts our assumption that $\lim\limits_{x\to 3} g(x)$ does not exist.

74. False, since the one-sided limits as $x \to 7$ from both sides are unequal.

75. True, since the one-sided limits as $x \to 7$ from both sides are equal to 4.

76. True, since the one-sided limits as $x \to 7$ from both sides exist, th ratio of their limits is equal to -1.

77. False, since we don't know if $f(x)$ or $(f(x))^2$ is defined at $x = 7$.

78. False, since $\lim_{x\to 7} f(x)$ does not exist.

79. True, since $\lim_{x\to 7}(f(x))^2 = 4$ and $(f(7)^2) = 4$.

80. True, since as $x \to \infty$, $1/x \to 0$ and as the sine function is continuous, $\sin(1/x) \to 0$, so the product of the two has a limit of 0.

81. False, since as $x \to 0^+$ the function $1/x \to \infty$. Also, as $x \to \infty$ the function $\sin(1/x)$ oscillates faster and faster between $+1$ and -1. Therefore the product of the two does not tend to either ∞ or $-\infty$ as $x \to 0$ on either side, but instead oscillates between large positive and large negative values.

Additional Problems (online only)

82. The function $f(x)$ is the product of the rational function $1/(x^2-1)$ with $e^{\sin(3x^2+2x-1)}$. The rational function is continuous so long as the denominator is not 0, that is, when $x \neq \pm 1$. The other function is a composition of a polynomial, a sine, and an exponential, so is continuous over any interval. Therefore the product of these two functions is continuous over any interval that does not contain $x = \pm 1$.

83. We have

$$\frac{x}{\sin x} = \frac{1}{(\sin x)/x},$$

so,

$$\lim_{x\to 0} \frac{x}{\sin x} = \frac{\lim_{x\to 0} 1}{\lim_{x\to 0}(\sin x)/x} = \frac{1}{1} = 1.$$

84. We have

$$\sin x = x\frac{\sin x}{x}.$$

Therefore,

$$\lim_{x\to 0} \sin x = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \frac{\sin x}{x}\right) = 0 \cdot 1 = 0.$$

85. We have $(\sin 3x)/x = 3(\sin 3x)/(3x)$. If we let $u = 3x$, then $u \to 0$ as $x \to 0$, so

$$\lim_{x\to 0} \frac{\sin 3x}{3x} = \lim_{u\to 0} \frac{\sin u}{u} = 1.$$

Thus

$$\lim_{x\to 0} \frac{\sin 3x}{x} = 3\lim_{x\to 0} \frac{\sin 3x}{3x} = 3.$$

86. We have

$$\frac{\cos^2 x - 1}{x^2} = \frac{-\sin^2 x}{x^2}.$$

Therefore

$$\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2} = -\left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{x}\right) = -1$$

87. (a) This statement follows: if we interchange the roles of f and g in the original statement, we get this statement.
(b) This statement is true, but it does not follow directly from the original statement, which says nothing about the case $g(a) = 0$.
(c) This follows, since if $g(a) \neq 0$ the original statement would imply f/g is continuous at $x = a$, but we are told it is not.
(d) This does not follow. Given that f is continuous at $x = a$ and $g(a) \neq 0$, then the original statement says g continuous implies f/g continuous, not the other way around. In fact, statement (d) is not true: if $f(x) = 0$ for all x, then g could be any discontinuous, non-zero function, and f/g would be zero, and therefore continuous. Thus the conditions of the statement would be satisfied, but not the conclusion.

Solutions for Section 1.9

Exercises

1. Canceling, we have

$$\lim_{x\to 0} \frac{3x^2}{x^2} = \lim_{x\to 0} 3 = 3.$$

2. Canceling, we have

$$\lim_{x\to 0} \frac{3x^2}{x} = \lim_{x\to 0} 3x = 0.$$

3. The limit does not exist since canceling, we have

$$\lim_{x\to 0}\frac{3x^2}{x^4}=\lim_{x\to 0}\frac{3}{x^2}=\infty.$$

4. Factoring the numerator and simplifying, we get

$$\frac{x^2-3x}{x-3}=\frac{x(x-3)}{x-3}=x$$

provided $x\neq 3$. Thus

$$\lim_{x\to 3}\frac{x^2-3x}{x-3}=\lim_{x\to 3}x=3.$$

5. Factoring the numerator and the denominator, we obtain

$$\frac{t^4+t^2}{2t^3-9t^2}=\frac{t^2(t^2+1)}{t^2(2t-9)}=\frac{t^2+1}{2t-9}$$

provided $t\neq 0$. Thus

$$\lim_{t\to 0}\frac{t^4+t^2}{2t^3-9t^2}=\lim_{t\to 0}\frac{t^2+1}{2t-9}=-\frac{1}{9}.$$

6. Factoring the numerator, we obtain

$$\frac{x^3-3x}{x\sqrt{2x+3}}=\frac{x(x^2-3)}{x\sqrt{2x+3}}=\frac{x^2-3}{\sqrt{2x+3}}$$

provided $x\neq 0$. Thus

$$\lim_{x\to 0}\frac{x^3-3x}{x\sqrt{2x+3}}=\lim_{x\to 0}\frac{x^2-3}{\sqrt{2x+3}}=-\frac{3}{\sqrt{3}}=-\sqrt{3}.$$

7. Factoring the denominator and simplifying, we get

$$\frac{x+4}{2x^2+5x-12}=\frac{x+4}{(x+4)(2x-3)}=\frac{1}{2x-3}$$

provided $x\neq -4$. Thus

$$\lim_{x\to -4}\frac{x+4}{2x^2+5x-12}=\lim_{x\to -4}\frac{1}{2x-3}=-\frac{1}{11}.$$

8. Factoring the numerator and simplifying, we get

$$\frac{y^2-5y+4}{y-1}=\frac{(y-1)(y-4)}{y-1}=y-4$$

provided $y\neq 1$. Thus

$$\lim_{y\to 1}\frac{y^2-5y+4}{y-1}=\lim_{y\to 1}(y-4)=-3.$$

9. Factoring the numerator and denominator and simplifying, we get

$$\frac{x^2+2x-3}{x^2-3x+2}=\frac{(x-1)(x+3)}{(x-1)(x-2)}=\frac{x+3}{x-2}$$

provided $x\neq 1$. Thus

$$\lim_{x\to 1}\frac{x^2+2x-3}{x^2-3x+2}=\lim_{x\to 1}\frac{x+3}{x-2}=-4.$$

10. Factoring the numerator and denominator and simplifying, we get

$$\frac{2t^2+3t-2}{t^2+5t+6} = \frac{(t+2)(2t-1)}{(t+2)(t+3)} = \frac{2t-1}{t+3}$$

provided $t \neq -2$. Thus

$$\lim_{t\to-2}\frac{2t^2+3t-2}{t^2+5t+6} = \lim_{t\to-2}\frac{2t-1}{t+3} = -5.$$

11. Factoring the numerator and the denominator and simplifying, we get

$$\frac{x^2-9}{x^2+x-12} = \frac{(x-3)(x+3)}{(x-3)(x+4)} = \frac{x+3}{x+4}$$

provided $x \neq 3$. Thus

$$\lim_{x\to3}\frac{x^2-9}{x^2+x-12} = \lim_{x\to3}\frac{x+3}{x+4} = \frac{6}{7}.$$

12. Factoring the numerator and the denominator, we obtain

$$\frac{2y^2+y-1}{3y^2+2y-1} = \frac{(2y-1)(y+1)}{(3y-1)(y+1)} = \frac{2y-1}{3y-1}$$

provided $y \neq -1$. Thus

$$\lim_{y\to-1}\frac{2y^2+y-1}{3y^2+2y-1} = \lim_{y\to-1}\frac{2y-1}{3y-1} = \frac{3}{4}.$$

13. Expanding the numerator, we get

$$(3+h)^2-9 = 9+6h+h^2-9 = 6h+h^2.$$

Next, factoring and simplifying, we get

$$\frac{(3+h)^2-9}{h} = \frac{h(6+h)}{h} = 6+h$$

provided $h \neq 0$. Thus

$$\lim_{h\to0}\frac{(3+h)^2-9}{h} = \lim_{h\to0}(6+h) = 6.$$

14. Simplifying and factoring the numerator and the denominator, we obtain

$$\frac{(x+5)^2-4}{x^2-9} = \frac{x^2+10x+21}{x^2-9} = \frac{(x+3)(x+7)}{(x+3)(x-3)} = \frac{x+7}{x-3}$$

provided $x \neq -3$. Thus

$$\lim_{x\to-3}\frac{(x+5)^2-4}{x^2-9} = \lim_{x\to-3}\frac{x+7}{x-3} = -\frac{2}{3}.$$

15. Factoring the numerator and the denominator, we obtain

$$\frac{5x^2-15}{x^4-9} = \frac{5(x^2-3)}{(x^2-3)(x^2+3)} = \frac{5}{x^2+3}$$

provided $x \neq \sqrt{3}$. Thus

$$\lim_{x\to\sqrt{3}}\frac{5x^2-15}{x^4-9} = \lim_{x\to\sqrt{3}}\frac{5}{x^2+3} = \frac{5}{6}.$$

16. Rewriting the numerator as a single quotient we get $2/x-1 = (2-x)/x$. Then

$$\frac{2/x-1}{x-2} = \frac{(2-x)/x}{x-2} = -\frac{1}{x}$$

provided $x \neq 2$. Thus

$$\lim_{x\to2}\frac{2/x-1}{x-2} = \lim_{x\to2}-\frac{1}{x} = -\frac{1}{2}.$$

17. Simplifying, we obtain

$$\frac{1/t-1/3}{t-3}=\frac{\frac{3-t}{3t}}{t-3}=\frac{3-t}{3t}\cdot\frac{1}{t-3}=\frac{-(t-3)}{3t(t-3)}=\frac{-1}{3t}$$

provided $t\neq 3$. Thus

$$\lim_{t\to 3}\frac{1/t-1/3}{t-3}=\lim_{t\to 3}\frac{-1}{3t}=-\frac{1}{9}.$$

18. Rewriting the numerator as a single quotient, we have

$$\frac{1}{t+1}-1=\frac{1-(t+1)}{t+1}=-\frac{t}{t+1}$$

Then

$$\frac{1/(t+1)-1}{t}=-\frac{t/(t+1)}{t}=-\frac{1}{t+1}$$

provided $t\neq 0$. Thus

$$\lim_{t\to 0}\frac{1/(t+1)-1}{t}=\lim_{t\to 0}-\frac{1}{t+1}=-1.$$

19. Rewriting the numerator as a single quotient we get

$$\frac{1}{4+h}-\frac{1}{4}=\frac{4}{4(4+h)}-\frac{4+h}{4(4+h)}=\frac{4-(4+h)}{4(4+h)}=\frac{-h}{4(4+h)}$$

provided $h\neq 0$. Thus

$$\lim_{h\to 0}\frac{1/(4+h)-1/4}{h}=\lim_{h\to 0}\frac{-h}{4h(4+h)}==\lim_{h\to 0}\frac{-1}{4(4+h)}=-\frac{1}{16}.$$

20. There are two ways to do this. One way is to multiply the numerator and denominator by $\sqrt{z}+1$ and then simplify:

$$\frac{(\sqrt{z}-1)(\sqrt{z}+1)}{(z-1)(\sqrt{z}+1)}=\frac{(\sqrt{z})^2-1}{(z-1)(\sqrt{z}+1)}=\frac{1}{\sqrt{z}+1}$$

provided $z\neq 1$. Thus

$$\lim_{z\to 1}\frac{\sqrt{z}-1}{z-1}=\lim_{z\to 1}\frac{1}{\sqrt{z}+1}=\frac{1}{2}.$$

A second way is to let $u=\sqrt{z}$ so $z=u^2$. Since $u\to 1$ as $z\to 1$, we can rewrite

$$\lim_{z\to 1}\frac{\sqrt{z}-1}{z-1}=\lim_{u\to 1}\frac{u-1}{u^2-1}.$$

Factoring $u^2-1=(u-1)(u+1)$ and canceling the $u-1$ factors since $u\neq 1$ in the limit, we get

$$\lim_{u\to 1}\frac{u-1}{u^2-1}=\lim_{u\to 1}\frac{1}{u+1}=\frac{1}{2}.$$

21. There are two ways to do this. One way is to multiply the numerator and denominator by $\sqrt{9+h}+3$ and then simplify:

$$\frac{\left(\sqrt{9+h}-3\right)\left(\sqrt{9+h}+3\right)}{h\left(\sqrt{9+h}+3\right)}=\frac{9+h-9}{h\left(\sqrt{9+h}+3\right)}=\frac{h}{h\left(\sqrt{9+h}+3\right)}=\frac{1}{\sqrt{9+h}+3}$$

provided $h\neq 0$. Thus

$$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}=\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}=\frac{1}{6}.$$

A second way is to let $u=\sqrt{9+h}$ so $h=u^2-9$. Since $u\to 3$ as $h\to 0$, we can rewrite

$$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}=\lim_{u\to 3}\frac{u-3}{u^2-9}.$$

Factoring $u^2 - 9 = (u-3)(u+3)$ and canceling the $u - 3$ factors since $u \neq 3$ in the limit, we get

$$\lim_{u\to 3} \frac{u-3}{u^2-9} = \lim_{u\to 3} \frac{1}{u+3} = \frac{1}{6}.$$

22. There are two ways to do this. First, factoring the numerator and simplifying, we get

$$\frac{4^x - 1}{2^x - 1} = \frac{(2^x)^2 - 1}{2^x - 1} = \frac{(2^x-1)(2^x+1)}{2^x - 1} = 2^x + 1$$

provided $x \neq 0$. Thus

$$\lim_{x\to 0} \frac{4^x - 1}{2^x - 1} = \lim_{x\to 0}(2^x + 1) = 2^0 + 1 = 2.$$

A second way is to let $u = 2^x$. Since $u \to 1$ as $x \to 0$, we can rewrite

$$\lim_{x\to 0} \frac{4^x - 1}{2^x - 1} = \lim_{u\to 1} \frac{u^2 - 1}{u - 1}.$$

Factoring $u^2 - 1 = (u-1)(u+1)$ and canceling the $u - 1$ factors since $u \neq 1$ in the limit, we get

$$\lim_{u\to 1} \frac{u^2-1}{u-1} = \lim_{u\to 1}(u+1) = 2.$$

23. Expanding the numerator, we get

$$(1+h)^4 - 1 = 1 + 4h + 6h^2 + 4h^3 + h^4 - 1.$$

Next, factoring and simplifying, we get

$$\frac{(1+h)^4 - 1}{h} = \frac{h(4 + 6h + 4h^2 + h^3)}{h} = 4 + 6h + 4h^2 + h^3$$

provided $h \neq 0$. Thus

$$\lim_{h\to 0} \frac{(1+h)^4 - 1}{h} = \lim_{h\to 0}(4 + 6h + 4h^2 + h^3) = 4.$$

24. (a) We have $f(4) = 7$ and $g(4) = 4^{-2}$, so the limit is of the form $7/(4^{-2})$, not $0/0$.
(b) Since $f(x)/g(x)$ is continuous at $c = 4$, we have

$$\lim_{x\to 4} \frac{f(x)}{g(x)} = \frac{f(4)}{g(4)} = \frac{7}{4^{-2}} = 7 \cdot 16 = 112.$$

25. (a) We have $f(1) = 2(1)^3 + (1)^2 - 3(1) = 0$ and $g(1) = (1)^2 + 3(1) - 4 = 0$, so the limit is of the form $0/0$.
(b) Factoring, we have

$$\begin{aligned}
\lim_{x\to 1} \frac{2x^3 + x^2 - 3x}{x^2 + 3x - 4} &= \lim_{x\to 1} \frac{x(2x^2 + x - 3)}{(x-1)(x+4)} \\
&= \lim_{x\to 1} \frac{x(2x+3)(x-1)}{(x-1)(x+4)} \\
&= \lim_{x\to 1} \frac{x(2x+3)}{(x+4)} \\
&= \lim_{x\to 1} \frac{(1)(2(1)+3)}{(1+4)} = \frac{5}{5} = 1.
\end{aligned}$$

So $\lim_{x\to 1} \frac{2x^3 + x^2 - 3x}{x^2 + 3x - 4} = 1.$

26. (a) We have $f(3) = (3)^2 - 3(3) = 0$ and $g(3) = \sqrt{3+1} - 2 = 0$, so the limit is of the form $0/0$.
(b) Rationalizing the denominator, we have

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-3x}{\sqrt{x+1}-2} &= \lim_{x\to 3} \frac{x^2-3x}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \\
&= \lim_{x\to 3} \frac{(x^2-3x)(\sqrt{x+1}+2)}{(x+1)-4} \\
&= \lim_{x\to 3} \frac{x(x-3)(\sqrt{x+1}+2)}{(x-3)} \\
&= \lim_{x\to 3} x(\sqrt{x+1}+2) \\
&= 3(\sqrt{3+1}+2) = 3(2+2) = 12.
\end{aligned}$$

So $\lim\limits_{x\to 3} \dfrac{x^2-3x}{\sqrt{x+1}-2} = 12.$

27. Dividing both numerator and denominator by z^3, we get

$$\lim_{z\to\infty} \frac{5z^2+2z+1}{2z^3-z^2+9} = \lim_{z\to\infty} \frac{\frac{5}{z}+\frac{2}{z^2}+\frac{1}{z^3}}{2-\frac{1}{z}+\frac{9}{z^3}} = \frac{0+0+0}{2-0+0} = 0.$$

28. Dividing both numerator and denominator by x^2, we get

$$\lim_{x\to\infty} \frac{x+7x^2-11}{3x^2-2x} = \lim_{x\to\infty} \frac{\frac{1}{x}+7-\frac{11}{x^2}}{3-\frac{2}{x}} = \frac{7}{3}.$$

29. Dividing both numerator and denominator by e^t, we get

$$\lim_{t\to\infty} \frac{4e^t+3e^{-t}}{5e^t+2e} = \lim_{t\to\infty} \frac{4+\frac{3}{e^{2t}}}{5+\frac{2e}{e^t}} = \frac{4}{5}.$$

30. We have,

$$\lim_{x\to\infty} \frac{2^{x+1}}{3^{x-1}} = \lim_{x\to\infty} \frac{2^x\cdot 2}{3^x\cdot 3^{-1}} = \lim_{x\to\infty} 6\left(\frac{2}{3}\right)^x = 0$$

since $\lim\limits_{x\to\infty}(2/3)^x = 0$ because $2/3 < 1$.

31. We have,

$$\lim_{x\to\infty} \frac{2^{3x+2}}{3^{x+3}} = \lim_{x\to\infty} \frac{2^{3x}\cdot 2^2}{3^x\cdot 3^3} = \lim_{x\to\infty} \left(\frac{2^3}{3}\right)^x\left(\frac{4}{27}\right) = \lim_{x\to\infty} \left(\frac{8}{3}\right)^x\left(\frac{4}{27}\right) = \infty,$$

since $\lim\limits_{x\to\infty}\left(\frac{8}{3}\right)^x = \infty$ because $\frac{8}{3} > 1$.

32. We have,

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = 0,$$

since e^x grows faster than x.

33. We have,

$$\lim_{t\to\infty}(4e^t)(7e^{-t}) = \lim_{t\to\infty} \frac{28e^t}{e^t} = 28.$$

34. We have,

$$\lim_{t\to\infty} t^{-2}\sin t = \lim_{t\to\infty} \frac{\sin t}{t^2} = 0,$$

since $-1 \le \sin t \le 1$ and $t^2 \to \infty$ as $t \to \infty$.

35. Given that $-4x+6 \le f(x) \le x^2-2x+7$, we see that

$$\begin{aligned}\lim_{x\to-1}(-4x+6) \le \lim_{x\to-1} f(x) &\le \lim_{x\to-1}(x^2-2x+7)\\ -4(-1)+6 \le \lim_{x\to-1} f(x) &\le (-1)^2-2(-1)+7\\ 10 \le \lim_{x\to-1} f(x) &\le 10.\end{aligned}$$

Since $10 \le \lim_{x\to-1} f(x) \le 10$, we know that $\lim_{x\to-1} f(x) = 10$.

36. Given that $4\cos(2x) \le f(x) \le 3x^2+4$, we see that

$$\begin{aligned}\lim_{x\to0} 4\cos(2x) \le \lim_{x\to0} f(x) &\le \lim_{x\to0}(3x^2+4)\\ 4\cos(0) \le \lim_{x\to0} f(x) &\le 3(0)^2+4\\ 4 \le \lim_{x\to0} f(x) &\le 4.\end{aligned}$$

Since $4 \le \lim_{x\to0} f(x) \le 4$, we know that $\lim_{x\to0} f(x) = 4$.

37. Given that $\dfrac{4x^2-5}{x^2} \le f(x) \le \dfrac{4x^6+3}{x^6}$, we see that

$$\begin{aligned}\lim_{x\to\infty}\frac{4x^2-5}{x^2} \le \lim_{x\to\infty} f(x) &\le \lim_{x\to\infty}\frac{4x^6+3}{x^6}\\ \lim_{x\to\infty}\frac{4x^2}{x^2}-\lim_{x\to\infty}\frac{5}{x^2} \le \lim_{x\to\infty} f(x) &\le \lim_{x\to\infty}\frac{4x^6}{x^6}+\lim_{x\to\infty}\frac{3}{x^6}\\ &\text{(using the sum property since each limit exists)}\\ 4-0 \le \lim_{x\to\infty} f(x) &\le 4+0\\ 4 \le \lim_{x\to\infty} f(x) &\le 4.\end{aligned}$$

Since $4 \le \lim_{x\to\infty} f(x) \le 4$, we know that $\lim_{x\to\infty} f(x) = 4$.

Problems

38. Because the denominator equals 0 when $x = 4$, so must the numerator. This means $k^2 = 16$ and the choices for k are 4 or -4.

39. Because the denominator equals 0 when $x = 1$, so must the numerator. So $1-k+4 = 0$. The only possible value of k is 5.

40. Because the denominator equals 0 when $x = -2$, so must the numerator. So $4-8+k = 0$ and the only possible value of k is 4.

41. Because the denominator equals 0 when $x = 5$, so must the numerator. So $25-5k+5 = 0$ and the only possible value of k is 6.

42. Because the denominator equals 0 when $x = 0$, so must the numerator. So $e^k - 8 = 0$ and the only possible value of k is $\ln 8$.

43. Because the denominator equals 0 when $x = 1$, so must the numerator. So $k^2 - 49 = 0$ and the choices for k are 7 or -7.

44. Division of numerator and denominator by x^2 yields

$$\frac{x^2+3x+5}{4x+1+x^k} = \frac{1+3/x+5/x^2}{4/x+1/x^2+x^{k-2}}.$$

As $x\to\infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k > 2$, the denominator approaches ∞ as $x\to\infty$, so the limit of the quotient is 0. If $k = 2$, the denominator approaches 1 as $x\to\infty$, so the limit of the quotient is 1. If $k < 2$ the denominator approaches 0^+ as $x\to\infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k \ge 2$.

45. For the numerator, $\lim_{x\to-\infty}\left(e^{2x}-5\right)=-5$. If $k>0$, $\lim_{x\to-\infty}\left(e^{kx}+3\right)=3$, so the quotient has a limit of $-5/3$. If $k=0$, $\lim_{x\to-\infty}\left(e^{kx}+3\right)=4$, so the quotient has limit of $-5/4$. If $k<0$, the limit of the quotient is given by $\lim_{x\to-\infty}(e^{2x}-5)/(e^{kx}+3)=0$.

46. Division of numerator and denominator by x^3 yields

$$\frac{x^3-6}{x^k+3}=\frac{1-6/x^3}{x^{k-3}+3/x^3}.$$

As $x\to\infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k>3$, the denominator approaches ∞ as $x\to\infty$, so the limit of the quotient is 0. If $k=3$, the denominator approaches 1 as $x\to\infty$, so the limit of the quotient is 1. If $k<3$ the denominator approaches 0^+ as $x\to\infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k\geq 3$.

47. We divide both the numerator and denominator by e^{5x}, giving

$$\lim_{x\to\infty}\frac{e^{kx}+11}{e^{5x}-3}=\lim_{x\to\infty}\frac{e^{(k-5)x}+11/e^{5x}}{1-3/e^{5x}}.$$

In the denominator, $\lim_{x\to\infty}1-3/e^{5x}=1$. In the numerator, if $k<5$, we have $\lim_{x\to\infty}(e^{(k-5)x}+11/e^{5x})=0$, so the quotient has a limit of 0. If $k=5$, we have $\lim_{x\to\infty}(e^{(k-5)x}+11/e^{5x})=1$, so the quotient has a limit of 1. If $k>5$, we have $\lim_{x\to\infty}(e^{(k-5)x}+11/e^{5x})=\infty$, so the limit of the quotient does not exist.

48. We divide both the numerator and denominator by 3^{2x}, giving

$$\lim_{x\to\infty}\frac{3^{kx}+6}{3^{2x}+4}=\frac{3^{(k-2)x}+6/3^{2x}}{1+4/3^{2x}}.$$

In the denominator, $\lim_{x\to\infty}1+4/3^{2x}=1$. In the numerator, if $k<2$, we have $\lim_{x\to\infty}3^{(k-2)x}+6/3^{2x}=0$, so the quotient has a limit of 0. If $k=2$, we have $\lim_{x\to\infty}3^{(k-2)x}+6/3^{2x}=1$, so the quotient has a limit of 1. If $k>2$, we have $\lim_{x\to\infty}3^{(k-2)x}+6/3^{2x}=\infty$, so the quotient has a limit of ∞.

49. In the denominator, we have $\lim_{x\to-\infty}3^{2x}+4=4$. In the numerator, if $k<0$, we have $\lim_{x\to-\infty}3^{kx}+6=\infty$, so the quotient has a limit of ∞. If $k=0$, we have $\lim_{x\to-\infty}3^{kx}+6=7$, so the quotient has a limit of $7/4$. If $k>0$, we have $\lim_{x\to-\infty}3^{kx}+6=6$, so the quotient has a limit of $6/4$.

50. Let $t=\sqrt{y}$ so that $y=t^2$. Since $t\to 2$ as $y\to 4$, we have

$$\begin{aligned}\lim_{y\to4}\frac{\sqrt{y}-2}{y-4}&=\lim_{t\to2}\frac{t-2}{t^2-4}\\&=\lim_{t\to2}\frac{t-2}{(t-2)(t+2)}\\&=\lim_{t\to2}\frac{1}{t+2}\\&=\frac{1}{4}.\end{aligned}$$

51. Let $t=\sqrt{x}$ so that $x=t^2$. Since $t\to 3$ as $x\to 9$, we have

$$\begin{aligned}\lim_{x\to9}\frac{x-\sqrt{x}-6}{\sqrt{x}-3}&=\lim_{t\to3}\frac{t^2-t-6}{t-3}\\&=\lim_{t\to3}\frac{(t-3)(t+2)}{t-3}\\&=\lim_{t\to3}(t+2)\\&=5.\end{aligned}$$

52. Let $t = \sqrt{1+h}$ so that $t^2 = 1 + h$ and $h = t^2 - 1$. Since $t \to 1$ as $h \to 0$, we have

$$\begin{aligned}\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} &= \lim_{t\to 1} \frac{t-1}{t^2-1}\\ &= \lim_{t\to 1} \frac{t-1}{(t-1)(t+1)}\\ &= \lim_{t\to 1} \frac{1}{t+1}\\ &= \frac{1}{2}.\end{aligned}$$

53. Let $t = \sqrt[3]{x}$ so that $x = t^3$. Since $t \to 1$ as $x \to 1$, we have

$$\begin{aligned}\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1} &= \lim_{t\to 1} \frac{t-1}{t^3-1}\\ &= \lim_{t\to 1} \frac{t-1}{(t-1)(t^2+t+1)}\\ &= \lim_{t\to 1} \frac{1}{t^2+t+1}\\ &= \frac{1}{3}.\end{aligned}$$

54. Let $t = e^x$ so that $e^{3x} = (e^x)^3 = t^3$ and $e^{2x} = t^2$. Since $t \to 1$ as $x \to 0$, we have

$$\begin{aligned}\lim_{x\to 0} \frac{e^{3x}-e^{2x}}{e^x-1} &= \lim_{t\to 1} \frac{t^3-t^2}{t-1}\\ &= \lim_{t\to 1} \frac{t^2(t-1)}{t-1}\\ &= \lim_{t\to 1} t^2\\ &= 1.\end{aligned}$$

55. Let $t = e^x$ so that $e^{3x} = (e^x)^3 = t^3$. Since $t \to \infty$ as $x \to \infty$, we have:

$$\begin{aligned}\lim_{x\to\infty} \frac{2e^{3x}-1}{5e^{3x}+e^x+1} &= \lim_{t\to\infty} \frac{2t^3-1}{5t^3+t+1}\\ &= \lim_{t\to\infty} \frac{2-1/t^3}{5+t/t^3+1/t^3} && \text{Dividing numerator and denominator by } t^3\\ &= \lim_{t\to\infty} \frac{2-1/t^3}{5+1/t^2+1/t^3} && \text{Simplifying}\\ &= \frac{2-0}{5+0+0} = \frac{2}{5}. && \text{Since } 1/t^2 \to 0 \text{ and } 1/t^3 \to 0 \text{ as } t \to \infty\end{aligned}$$

56. We know that $-1 \le \sin x \le 1$. If we divide by x, $x \neq 0$, we get $-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$. Since

$$\lim_{x\to\infty} -\frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x},$$

we know that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

57. For $x \ge 0$, since $0 < e^{-x} \le 1$, we have $x < x + e^{-x} < x + 1$. Therefore, $\frac{1}{x+1} < \frac{1}{x+e^{-x}} < \frac{1}{x}$. Therefore, since $\lim_{x\to\infty} \frac{1}{x+1} = 0 = \lim_{x\to\infty} \frac{1}{x}$, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x+1} \le \lim_{x\to\infty} &\frac{1}{x+e^{-x}} \le \lim_{x\to\infty} \frac{1}{x}\\ 0 \le \lim_{x\to\infty} &\frac{1}{x+e^{-x}} \le 0\end{aligned}$$

Thus, $\lim_{x\to\infty} \dfrac{1}{x+e^{-x}} = 0$.

58. For all x, we know that $-1 \le \cos x \le 1$, and hence that $0 \le \cos^2 x \le 1$. It follows that

$$0 \le \frac{\cos^2 x}{2x+1} \le \frac{1}{2x+1}.$$

Therefore, since $\lim_{x\to\infty} 0 = 0$ and $\lim_{x\to\infty} \dfrac{1}{2x+1} = 0$, we have $\lim_{x\to\infty} \dfrac{\cos^2 x}{2x+1} = 0$ by the Squeeze Theorem.

59. For all $x \ne 0$, we know that $-1 \le \sin(1/x) \le 1$. It follows that

$$-x^4 \le x^4 \sin(1/x) \le x^4.$$

Therefore, since $\lim_{x\to 0}(-x^4) = 0$ and $\lim_{x\to 0} x^4 = 0$, we have $\lim_{x\to 0} x^4 \sin(1/x) = 0$ by the Squeeze Theorem.

60. For all $x \ge 0$, we know that

$$0 \le \frac{x}{\sqrt{x^3+1}} \le \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}.$$

Therefore, since $\lim_{x\to\infty} 0 = 0$ and $\lim_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$, we have $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^3+1}} = 0$ by the Squeeze Theorem.

61. We know that $-1 \le \cos x \le 1$, and hence that $0 \le 2\cos^2 x \le 2$. Therefore, $x \le x + 2\cos^2 x \le x+2$, so for $x > 0$,

$$\frac{1}{x+2} \le \frac{1}{x+2\cos^2 x} \le \frac{1}{x}.$$

Since $\lim_{x\to\infty} \dfrac{1}{x+2} = 0$ and $\lim_{x\to\infty} \dfrac{1}{x} = 0$, we have $\lim_{x\to\infty} \dfrac{1}{x+2\cos^2 x} = 0$ by the Squeeze Theorem.

62. There are many possible correct answers.

(a) If $f(x) = 1/x^2$ and $g(x) = 1/x$ then $f(x)/g(x) = 1/x$ so $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$.

(b) If $f(x) = 1/x$ and $g(x) = 1/x$ then $f(x)/g(x) = 1$ so $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$.

(c) If $f(x) = 1/x$ and $g(x) = 1/x^2$ then $f(x)/g(x) = x$ so $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$.

63. There are many possible choices for $g(x)$. One possibility is

$$g(x) = \left(\frac{x-3}{x-3}\right)\cdot f(x) = \left(\frac{x-3}{x-3}\right)(x^2+1) = \frac{x^3-3x^2+x-3}{x-3}.$$

Observe that $f(x) = g(x)$ wherever $f(x)$ is defined except for $x = 3$ since $g(3)$ is undefined. This means the graphs of $f(x)$ and $g(x)$ will be the same except at $x = 3$ where the graph of $g(x)$ has a hole.

Since the graphs of $f(x)$ and $g(x)$ are identical except at $x = 3$, the hole will have the same coordinates as the function $f(x)$ at $x = 3$, so the coordinates of the hole will be $(3, 10)$.

64. There are many possible choices for $g(x)$. One possibility is

$$g(x) = \left(\frac{x}{x}\right)\cdot f(x) = \left(\frac{x}{x}\right)(x^2+1) = \frac{x^3+x}{x}.$$

Observe that $f(x) = g(x)$ wherever $f(x)$ is defined except for $x = 0$ since $g(0)$ is undefined. This means the graphs of $f(x)$ and $g(x)$ will be the same except at $x = 0$ where the graph of $g(x)$ has a hole.

Since the graphs of $f(x)$ and $g(x)$ are identical except at $x = 0$, the hole will have the same coordinates as the function $f(x)$ at $x = 0$, so the coordinates of the hole will be $(0, 1)$.

65. There are many possible choices for $g(x)$. One possibility is

$$g(x) = \left(\frac{x-1}{x-1}\right)\cdot f(x) = \left(\frac{x-1}{x-1}\right)\ln x = \frac{x\ln x - \ln x}{x-1}.$$

Observe that $f(x) = g(x)$ wherever $f(x)$ is defined except for $x = 1$ since $g(1)$ is undefined. This means the graphs of $f(x)$ and $g(x)$ will be the same except at $x = 1$ where the graph of $g(x)$ has a hole.

Since the graphs of $f(x)$ and $g(x)$ are identical except at $x = 1$, the hole will have the same coordinates as the function $f(x)$ at $x = 1$, so the coordinates of the hole will be $(1, 0)$.

66. There are many possible choices for $g(x)$. One possibility is

$$g(x) = \left(\frac{x-\pi}{x-\pi}\right) \cdot f(x) = \left(\frac{x-\pi}{x-\pi}\right) \sin x = \frac{x \sin x - \pi \sin x}{x - \pi}.$$

Observe that $f(x) = g(x)$ wherever $f(x)$ is defined except for $x = \pi$ since $g(\pi)$ is undefined. This means the graphs of $f(x)$ and $g(x)$ will be the same except at $x = \pi$ where the graph of $g(x)$ has a hole.

Since the graphs of $f(x)$ and $g(x)$ are identical except at $x = \pi$, the hole will have the same coordinates as the function $f(x)$ at $x = \pi$, so the coordinates of the hole will be $(\pi, 0)$.

67. We have

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{(x-1)^3}{(x-1)^2} \\ &= \lim_{x\to 1}(x-1) && \text{Cancelling } (x-1) \text{ since } x \neq 1 \\ &= 1 - 1 = 0. && \text{Substituting } x = 1 \text{ since } (x-1) \text{ is continuous}\end{aligned}$$

68. We have

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{(x-1)^2}{(x-1)^3} \\ &= \lim_{x\to 1} \frac{1}{(x-1)} && \text{Cancelling } (x-1)^2 \text{ since } x \neq 1\end{aligned}$$

So the function $f(x)$ takes the same values as $1/(x-1)$ except at $x = 1$, so it must have the same limit. Since the limit of the denominator of $1/(x-1)$ is 0 and the numerator is 1 at $x = 1$, this limit does not exist.

69. We have

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{(x-1)^2}{(x-1)^2} \\ &= \lim_{x\to 1}(1) && \text{Cancelling } (x-1)^2 \text{ since } x \neq 1 \\ &= 1.\end{aligned}$$

70. We have

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{(x-1)^n}{(x-1)^m} \\ &= \lim_{x\to 1}(x-1)^{n-m} && \text{Cancelling } (x-1)^m \text{ since } x \neq 1 \text{ and } n > m \\ &= (1-1)^{n-m} = 0. && \text{Substituting } x = 1 \text{ since } (x-1)^{n-m} \text{ is continuous}\end{aligned}$$

71. We have

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{(x-1)^n}{(x-1)^m} \\ &= \lim_{x\to 1} \frac{1}{(x-1)^{m-n}} && \text{Cancelling } (x-1)^n \text{ since } x \neq 1 \text{ and } m > n\end{aligned}$$

So the function $f(x)$ takes the same values as $1/(x-1)^{m-n}$ except at $x = 1$, so it must have the same limit. Since the limit of the denominator of $1/(x-1)^{m-n}$ is 0 and the numerator is 1 at $x = 1$, this limit does not exist.

72. Since $m = n$, we have

$$\begin{aligned}\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{(x-1)^n}{(x-1)^n} \\ &= \lim_{x\to 1}(1) && \text{Cancelling } (x-1)^n \text{ since } x \neq 1 \\ &= 1.\end{aligned}$$

73. The Squeeze Theorem can be applied when $c = \infty$ or $c = -\infty$ since $\lim_{x\to\infty} -\frac{1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} -\frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}$. In both cases $L = 0$. The Squeeze Theorem cannot be applied to other cases since for any other c,

$$\lim_{x\to c} -\frac{1}{x} = -\frac{1}{c} \neq \frac{1}{c} = \lim_{x\to c} \frac{1}{x}.$$

Strengthen Your Understanding

74. If we factor the numerator of f and simplify, we see that

$$f(x) = x - 1, \text{ for } x \neq -1.$$

This show that the two functions are identical (that is, have the same values) where they are both defined, the functions have different domains: $x = -1$ is in the domain of g but not in the domain of f.

75. The limit $\lim_{x\to 1} f(x)/g(x)$ depends on the values of $f(x)/g(x)$ as x approaches 1 but is not necessarily dependent on the value at $x = 1$. For example, we can take $g(x) = 1$ and

$$f(x) = \begin{cases} -1 & x < 1 \\ 0 & x = 1 \\ 1 & 1 < x \end{cases}.$$

For these choices of $f(x)$ and $g(x)$, we have $f(1) = 0$ and $g(1) = 1$ as required. However, for all $x < 1$, $f(x)/g(x) = -1$ so $\lim_{x\to 1^-} f(x)/g(x) = -1$, and for $x > 1$, $f(x)/g(x) = 1$, so $\lim_{x\to 1^+} f(x)/g(x) = 1$. Thus the one sided limits are different, so $\lim_{x\to 1} f(x)/g(x)$ does not exist.

76. False, since the limit $\lim_{x\to 0} f(x)$ may not even exist. For example if $b(x) = -1, a(x) = 1$ and $f(x) = \sin(1/x)$, then the conditions are met but $\sin(1/x)$ has no limit as $x \to 0$.

77. True, by the Squeeze Theorem, using the constant function $b(x) = 0$.

78. False, this is a confused, and incorrect, version of the Squeeze Theorem. For example if $b(x) = f(x) = x$, and $a(x) = 1$ then the conditions are met but $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} a(x) = 1$.

79. False, for example if $f(x) = 0$, and $g(x) = x$ then the conditions are met but $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.

80. False, for example if $f(x) = 1$, and $g(x) = x$ then the conditions are met but $\lim_{x\to 0} \frac{f(x)}{g(x)}$ does not exist, since $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

81. False, for example if $f(x) = 1/x$, and $g(x) = 1/x^2$ then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} x = 0$ but the limits of $f(x)$ and $g(x)$ do not exist, since $\lim_{x\to 0^+} \frac{1}{x} = \infty$ (and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$) and $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

82. True, this follows from the multiplicative limit property: if $\lim_{x\to 0} a(x)$ exists and $\lim_{x\to 0} b(x)$ exists then $\lim_{x\to 0} a(x)\cdot b(x)$ exists (using $a(x) = f(x)/g(x)$ and $b(x) = g(x)$).

83. False, for example if $g(x) = 1$ when $x > 0$ and $g(x) = -1$ when $x < 0$, then setting $f(x) = g(x)$, and using $c = 0$ gives $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} 1 = 1$.

Solutions for Section 1.10

Exercises

1. For each δ we determine whether the graph of $f(x)$ on the interval $3 - \delta < x < 3 + \delta$ stays in the shaded region $L - \epsilon < y < L + \epsilon$ in the figure. Notice that as δ gets smaller, the interval gets shorter.

(a) No for $\delta = 1$. The graph of $f(x)$ on the interval $2 < x < 4$ extends outside the shaded region.

(b) No for $\delta = 0.75$. The graph of $f(x)$ on the interval $2.25 < x < 3.75$ extends outside the shaded region.
(c) Yes for $\delta = 0.5$. The graph of $f(x)$ on the interval $2.5 < x < 3.5$ fits inside the shaded region.
(d) Yes for $\delta = 0.25$. The graph of $f(x)$ on the interval $2.75 < x < 3.25$ fits inside the shaded region.
(e) Yes for $\delta = 0.1$. The graph of $f(x)$ on the interval $2.9 < x < 3.1$ fits inside the shaded region.

2. For each δ we determine whether the graph of $g(x)$ on the interval $20 - \delta < x < 20 + \delta$ stays in the shaded region $L - \epsilon < y < L + \epsilon$ in the figure. Notice that as δ gets smaller, the interval gets shorter.

(a) No for $\delta = 0.5$. The graph of $g(x)$ on the interval $19.5 < x < 20.5$ extends outside the shaded region.
(b) No for $\delta = 0.4$. The graph of $gx)$ on the interval $19.6 < x < 20.4$ extends outside the shaded region.
(c) No for $\delta = 0.3$. The graph of $g(x)$ on the interval $19.7 < x < 20.3$ extends outside the shaded region.
(d) Yes for $\delta = 0.2$. The graph of $g(x)$ on the interval $19.8 < x < 20.2$ fits inside the shaded region.
(e) Yes for $\delta = 0.1$. The graph of $g(x)$ on the interval $19.9 < x < 20.1$ fits inside the shaded region.

3. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.2$, $\delta \le 0.1$.
For $\epsilon = 0.1$, $\delta \le 0.05$.
For $\epsilon = 0.02$, $\delta \le 0.01$.
For $\epsilon = 0.01$, $\delta \le 0.005$.
For $\epsilon = 0.002$, $\delta \le 0.001$.
For $\epsilon = 0.001$, $\delta \le 0.0005$.

4. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \le 0.46$.
For $\epsilon = 0.01$, $\delta \le 0.21$.
For $\epsilon = 0.001$, $\delta < 0.1$. Thus, we can take $\delta \le 0.09$.

5. Since $|4x - 12| = 4|x - 3|$, to get $|4x - 12| < 0.1$ we require that $|x - 3| < 0.1/4 = 0.025$. Thus we take $\delta = 0.025$.

6. Since $|(2x-2)-0| = 2|x-1|$, to get $|(2x-2)-0| < 0.1$ we require that $|x-1| < 0.1/2 = 0.05$. Thus we take $\delta = 0.05$.

7. Since $|(5x + 3) - (-7)| = |5x + 10| = 5|x + 2|$, to get $|(5x + 3) - (-7)| < 0.1$ we require that $|x + 2| < 0.1/5 = 0.02$. Thus we take $\delta = 0.02$.

8. Since $|(x^2+2)-2| = |x^2| = |x|^2$, to get $|(x^2+2)-2| < 0.1$ we require that $|x| < \sqrt{0.1} = 0.316$. Thus we take $\delta = 0.316$.

Problems

9. The statement

$$\lim_{h\to a} g(h) = K$$

means that we can make the value of $g(h)$ as close to K as we want by choosing h sufficiently close to, but not equal to, a.

In symbols, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|g(h) - K| < \epsilon \quad \text{for all } 0 < |h - a| < \delta.$$

10. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \le 0.1$
For $\epsilon = 0.05$, $\delta \le 0.05$.
For $\epsilon = 0.0007$, $\delta \le 0.00007$.

11. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \le 0.45$.
For $\epsilon = 0.001$, $\delta \le 0.0447$.
For $\epsilon = 0.00001$, $\delta \le 0.00447$.

12. We want to find $\delta > 0$ such that if $-\delta < x < \delta$, then $|e^x - 1| < 0.1$. Observe that

$$\begin{aligned} |e^x - 1| &< 0.1 \\ -0.1 < e^x - 1 &< 0.1 \\ \ln(0.9) < x &< \ln(1.1), \end{aligned}$$

and that $\ln(0.9) = -0.105$ and $\ln(1.1) = 0.095$. We therefore choose $\delta = \ln(1.1)$ so that $-\delta < x < \delta$ implies

$$\ln(0.9) < -\ln(1.1) < x < \ln(1.1),$$

which, in turn, implies that $|e^x - 1| < 0.1$ (see Figure 1.118).

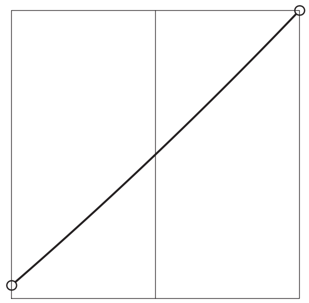


Figure 1.118: Graph of $e^x - 1$ with $-\ln(1.1) < x < \ln(1.1)$

13. We want to find $\delta > 0$ such that if $-\delta < x - 1 < \delta$, then $|\ln x| < 0.1$. Observe that

$$\begin{aligned} |\ln x| &< 0.1 \\ -0.1 < \ln x &< 0.1 \\ e^{-0.1} < x &< e^{0.1} \\ e^{-0.1} - 1 < x - 1 &< e^{0.1} - 1 \end{aligned}$$

and that $1 - e^{-0.1} = 0.095$ and $e^{0.1} - 1 = 0.105$. We therefore choose $\delta = 1 - e^{-0.1}$ so that $-\delta < x - 1 < \delta$ implies

$$e^{-0.1} - 1 < x - 1 < 1 - e^{-0.1} < e^{0.1} - 1,$$

which, in turn, implies that $|\ln x| < 0.1$ (see Figure 1.119).

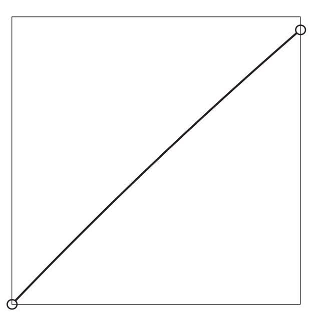


Figure 1.119: Graph of $\ln x$ with $1 - \delta < x < 1 + \delta$, where $\delta = 1 - e^{-0.1}$.

14. We want to find $\delta > 0$ such that if $-\delta < x - 1 < \delta$, then $|\arctan x - (\pi/4)| < 0.1$. Observe that

$$\begin{aligned} \left|\arctan x - \frac{\pi}{4}\right| &< 0.1 \\ -0.1 < \arctan x - \frac{\pi}{4} &< 0.1 \\ \frac{\pi}{4} - 0.1 < \arctan x &< \frac{\pi}{4} + 0.1 \\ \tan\left(\frac{\pi}{4} - 0.1\right) < x &< \tan\left(\frac{\pi}{4} + 0.1\right) \\ \tan\left(\frac{\pi}{4} - 0.1\right) - 1 < x - 1 &< \tan\left(\frac{\pi}{4} + 0.1\right) - 1, \end{aligned}$$

and that $1 - \tan((\pi/4) - 0.1) = 0.182$ and $\tan((\pi/4) + 0.1) - 1 = 0.223$. We therefore choose $\delta = 1 - \tan((\pi/4) - 0.1)$ so that $-\delta < x - 1 < \delta$ implies

$$\tan\left(\frac{\pi}{4} - 0.1\right) - 1 < x - 1 < 1 - \tan\left(\frac{\pi}{4} - 0.1\right) < \tan\left(\frac{\pi}{4} + 0.1\right) - 1,$$

which, in turn, implies that $|\arctan x - (\pi/4)| < 0.1$ (see Figure 1.120).

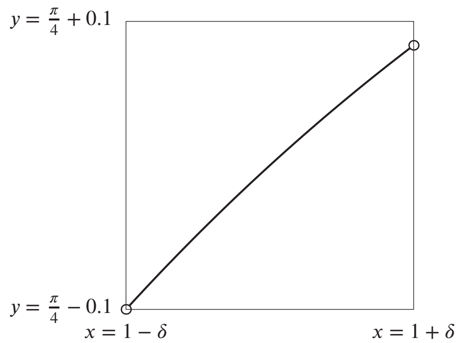


Figure 1.120: Graph of $\arctan x$ with $1 - \delta < x < 1 + \delta$, where $\delta = 1 - \tan((\pi/4) - 0.01)$.

15. We want to find $\delta > 0$ such that if $-\delta < x - 2 < \delta$, then $|(1/x) - (1/2)| < 0.1$. Observe that

$$\begin{aligned}
\left|\frac{1}{x} - \frac{1}{2}\right| &< 0.1 \\
\frac{1}{2} - 0.1 < \frac{1}{x} &< \frac{1}{2} + 0.1 \\
\frac{2}{5} < \frac{1}{x} &< \frac{3}{5} \\
\frac{5}{3} < x &< \frac{5}{2} \\
-\frac{1}{3} < x - 2 &< \frac{1}{2}.
\end{aligned}$$

We therefore choose $\delta = 1/3$ so that $-\delta < x - 2 < \delta$ implies

$$-\frac{1}{3} < x - 2 < \frac{1}{3} < \frac{1}{2},$$

which, in turn, implies that $|(1/x) - (1/2)| < 0.1$ (see Figure 1.121).

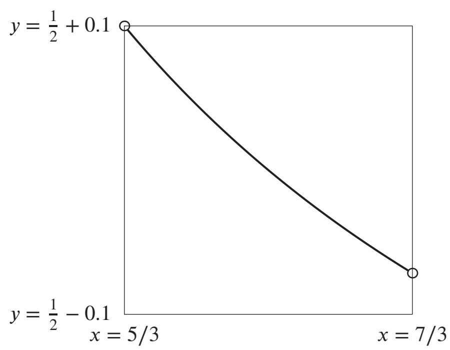


Figure 1.121: Graph of $1/x$ with $5/3 < x < 7/3$.

16. We want to find $\delta > 0$ such that if $-\delta < x - 4 < \delta$, then $|\sqrt{x} - 2| < 0.1$. Observe that

$$\begin{aligned} |\sqrt{x} - 2| &< 0.1 \\ 1.9 < \sqrt{x} &< 2.1 \\ 3.61 < x &< 4.41 \\ -0.39 < x - 4 &< 0.41. \end{aligned}$$

We therefore choose $\delta = 0.39$ so that $-\delta < x - 4 < \delta$ implies

$$-0.39 < x - 4 < 0.39 < 0.41,$$

which, in turn, implies that $|\sqrt{x} - 2| < 0.1$ (see Figure 1.122).

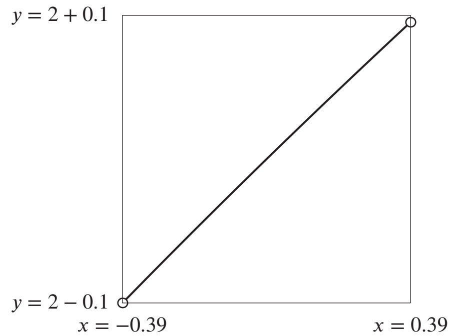


Figure 1.122: Graph of $\sqrt{x}$ with $-0.39 < x < 0.39$.

17. From Table 1.7, it appears the limit is 0. This is confirmed by Figure 1.123. An appropriate window is $-0.015 < x < 0.015$, $-0.01 < y < 0.01$.

Table 1.7

x	$f(x)$
0.1	0.0666
0.01	0.0067
0.001	0.0007
0.0001	0.0001

x	$f(x)$
−0.0001	−0.0001
−0.001	−0.0007
−0.01	−0.0067
−0.1	−0.0666

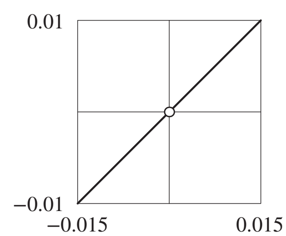


Figure 1.123

18. From Table 1.8, it appears the limit is 0. This is confirmed by Figure 1.124. An appropriate window is $-0.0029 < x < 0.0029$, $-0.01 < y < 0.01$.

Table 1.8

x	$f(x)$
0.1	0.3365
0.01	0.0337
0.001	0.0034
0.0001	0.0004

x	$f(x)$
−0.0001	−0.0004
−0.001	−0.0034
−0.01	−0.0337
−0.1	−0.3365

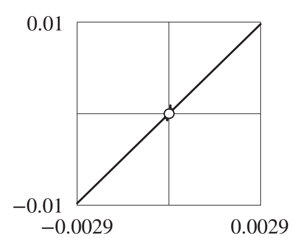


Figure 1.124

19. For any $\epsilon > 0$, we must show that there exists $\delta > 0$ such that, if $|x-2| < \delta$ with $x \neq 2$, then $|(5x-6)-4| < \epsilon$. Observe that

$$|(5x-6)-4| = |5x-10| = 5|x-2|.$$

Choose $\delta = \epsilon/5$. If $|x-2| < \delta$ with $x \neq 2$, we have

$$|(5x-6)-4| = 5|x-2| < 5\delta = \epsilon,$$

proving that $\lim_{x\to 2}(5x-6) = 4$.

20. For any $\epsilon > 0$, we must show that there exists $\delta > 0$ such that, if $|x+1| < \delta$ with $x \neq -1$, then $|(3x+1)+2| < \epsilon$. Observe that

$$|(3x+1)+2| = |3x+3| = 3|x+1|.$$

Choose $\delta = \epsilon/3$. If $|x+1| < \delta$ with $x \neq -1$, we have

$$|(3x+1)+2| = 3|x+1| < 3\delta = \epsilon,$$

proving that $\lim_{x\to -1}(3x+1) = -2$.

21. For any $\epsilon > 0$, we must show that there exists $\delta > 0$ such that, if $|x| < \delta$ with $x \neq 0$, then $|(-2x+3)-3| < \epsilon$. Observe that

$$|(-2x+3)-3| = |-2x| = 2|x|.$$

Choose $\delta = \epsilon/2$. If $|x| < \delta$ with $x \neq 0$, we have

$$|(-2x+3)-3| = 2|x| < 2\delta = \epsilon,$$

proving that $\lim_{x\to 0}(-2x+3) = 3$.

22. For any $\epsilon > 0$, we must show that there exists $\delta > 0$ such that, if $|x| < \delta$ with $x \neq 0$, then $|(x^2+2)-2| < \epsilon$. Observe that

$$|(x^2+2)-2| = |x^2| = |x|^2.$$

Choose $\delta = \sqrt{\epsilon}$. If $|x| < \delta$ with $x \neq 0$, we have

$$|(x^2+2)-2| = |x|^2 < \delta^2 = \epsilon,$$

proving that $\lim_{x\to 0}(x^2+2) = 2$.

23. For any $\epsilon > 0$, we want to find the δ such that

$$|g(x)-2| = \left|-x^3+2-2\right| = \left|x^3\right| < \epsilon.$$

Choose $\delta = \epsilon^{1/3}$. Then if $|x| < \delta = \epsilon^{1/3}$, it follows that $|g(x)-2| = \left|x^3\right| < \epsilon$.

24. For any $\epsilon > 0$, we must show that there exists $\delta > 0$ such that, if $|x-1| < \delta$ with $x \neq -1$, then

$$\left|\frac{2x^2+x-3}{x-1} - 5\right| < \epsilon.$$

Observe that, if $x \neq 1$, then

$$\left|\frac{2x^2+x-3}{x-1} - 5\right| = \left|\frac{(2x+3)(x-1)}{x-1} - 5\right| = |2x+3-5| = 2|x-1|.$$

Choose $\delta = \epsilon/2$. If $|x-1| < \delta$ with $x \neq 1$, we have

$$\left|\frac{2x^2+x-3}{x-1} - 5\right| = 2|x-1| < 2\delta = \epsilon,$$

proving that $\lim_{x\to 1}\dfrac{2x^2+x-3}{x-1} = 5$.

25. For any $\epsilon > 0$, we must show that there exists $\delta > 0$ such that, if $|x+3| < \delta$ with $x \neq -3$, then

$$\left|\frac{x^2+2x-3}{x+3}+4\right| < \epsilon.$$

Observe that, if $x \neq -3$, then

$$\left|\frac{x^2+2x-3}{x+3}+4\right| = \left|\frac{(x+3)(x-1)}{x+3}+4\right| = |x+3|.$$

Choose $\delta = \epsilon$. If $|x+3| < \delta$ with $x \neq -3$, we have

$$\left|\frac{x^2+2x-3}{x+3}+4\right| = |x+3| < \delta = \epsilon,$$

proving that $\lim\limits_{x\to -3} \dfrac{x^2+2x-3}{x+3} = -4$.

26. (a) We want to show that, no matter how close x gets to 0, there will always be values of x such that $f(x)$ is not close to 1. By choosing values of x for which $f(x) = 0$, we see that $|f(x) - 1| = 1$; that is, the distance between $f(x)$ and 1 equals 1. For any nonzero integer n, we have

$$\begin{aligned} f(x) &= 0 \\ \sin(1/x) &= 0 \\ \frac{1}{x} &= n\pi \\ x &= \frac{1}{n\pi}, \end{aligned}$$

so $f(1/(n\pi)) = 0$. Choose ϵ to be any real number less than one, and consider any $\delta > 0$. Then we can choose a positive integer n large enough so that, if $x = 1/(n\pi)$, then $|x| < \delta$. Thus, x is close to 0, but $f(x) = 0$ is not close to 1 because $|f(x) - 1| = 1 > \epsilon$ (see Figure 1.125). This proves that $\lim\limits_{x\to 0} f(x) \neq 1$.

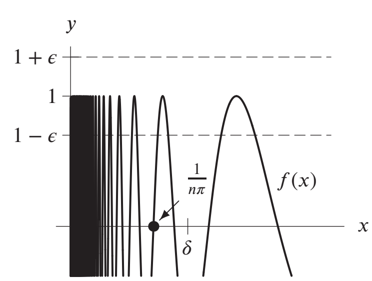


Figure 1.125

(b) We want to show that, no matter how close x gets to 0, there will always be values of x such that $f(x)$ is not close to L. By choosing values of x for which $f(x) = 1$, we see that $|f(x) - L| = |1 - L|$; that is, the distance between $f(x)$ and L equals $|1 - L| > 0$. Observe that

$$\begin{aligned} f(x) &= 1 \\ \sin(1/x) &= 1 \\ \frac{1}{x} &= \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \dots \\ x &= \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \frac{2}{13\pi}, \dots, \end{aligned}$$

so $f(x) = 0$ if $x = 2/((4n-3)\pi)$ for a positive integer n. Choose $\epsilon = |1 - L|/2$, and consider any $\delta > 0$. Then we can choose a positive integer n large enough so that, if $x = 2/((4n-3)\pi)$, then $|x| < \delta$. Thus, x is close to 0, but $f(x) = 1$ is not close to L because $|f(x) - L| = |1 - L| > \epsilon$ (see Figure 1.126). This proves that $\lim\limits_{x\to 0} f(x) \neq L$.

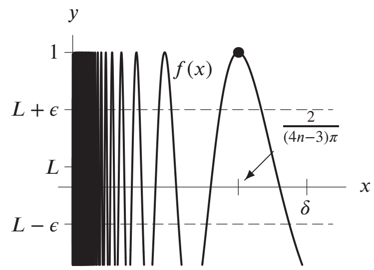


Figure 1.126

27. (a) For any c, we have $f(c) = k$. This means $|f(x) - f(c)| = |k - k| = 0$ for any x and c. So given $\epsilon > 0$, choose any δ, for example $\delta = 1$. Then $|x - c| < \delta$ implies $|f(x) - f(c)| = 0 < \epsilon$. So $\lim_{x\to c} f(x) = f(c)$, so f is continuous everywhere.

(b) Given any $\epsilon > 0$, we can take $\delta = \epsilon$, so that when $|x - c| < \delta$, $|g(x) - g(c)| = |x - c| < \epsilon$. So, $\lim_{x\to c} g(x) = g(c)$ and g is continuous everywhere.

28. The only change is that, instead of considering all x near c, we only consider x near to and greater than c. Thus the phrase "$|x - c| < \delta$" must be replaced by "$c < x < c + \delta$." Thus, we define

$$\lim_{x\to c^+} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c < x < c + \delta$, then $|f(x) - L| < \epsilon$.

29. The only change is that, instead of considering all x near c, we only consider x near to and less than c. Thus the phrase "$|x - c| < \delta$" must be replaced by "$c - \delta < x < c$." Thus, we define

$$\lim_{x\to c^-} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c - \delta < x < c$, then $|f(x) - L| < \epsilon$.

30. Instead of being "sufficiently close to c," we want x to be "sufficiently large." Using N to measure how large x must be, we define

$$\lim_{x\to\infty} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $N > 0$ (sufficiently large) such that if $x > N$, then $|f(x) - L| < \epsilon$.

31. (a) If $b = 0$, then the property says $\lim_{x\to c} 0 = 0$, which is easy to see is true.

(b) If $|f(x) - L| < \frac{\epsilon}{|b|}$, then multiplying by $|b|$ gives

$$|b||f(x) - L| < \epsilon.$$

Since

$$|b||f(x) - L| = |b(f(x) - L)| = |bf(x) - bL|,$$

we have

$$|bf(x) - bL| < \epsilon.$$

(c) Suppose that $\lim_{x\to c} f(x) = L$. We want to show that $\lim_{x\to c} bf(x) = bL$. If we are to have

$$|bf(x) - bL| < \epsilon,$$

then we will need

$$|f(x) - L| < \frac{\epsilon}{|b|}.$$

We choose δ small enough that

$$|x - c| < \delta \quad \text{implies} \quad |f(x) - L| < \frac{\epsilon}{|b|}.$$

By part (b), this ensures that

$$|bf(x) - bL| < \epsilon,$$

as we wanted.

32. Suppose $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$. Then we need to show that

$$\lim_{x\to c}(f(x)+g(x)) = L_1 + L_2.$$

Let $\epsilon > 0$ be given. We need to show that we can choose $\delta > 0$ so that whenever $|x-c| < \delta$, we will have $\left|(f(x)+g(x)) - (L_1+L_2)\right| < \epsilon$. First choose $\delta_1 > 0$ so that $|x-c| < \delta_1$ implies $|f(x)-L_1| < \frac{\epsilon}{2}$; we can do this since $\lim_{x\to c} f(x) = L_1$. Similarly, choose $\delta_2 > 0$ so that $|x-c| < \delta_2$ implies $|g(x)-L_2| < \frac{\epsilon}{2}$. Now, set δ equal to the smaller of δ_1 and δ_2. Thus $|x-c| < \delta$ will make both $|x-c| < \delta_1$ and $|x-c| < \delta_2$. Then, for $|x-c| < \delta$, we have

$$\begin{aligned}
\left|f(x)+g(x)-(L_1+L_2)\right| &= \left|\left(f(x)-L_1\right) + \left(g(x)-L_2\right)\right| \\
&\le \left|\left(f(x)-L_1\right)\right| + \left|\left(g(x)-L_2\right)\right| \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

This proves $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$, which is the result we wanted to prove.

33. (a) We need to show that for any given $\epsilon > 0$, there is a $\delta > 0$ so that $|x-c| < \delta$ implies $|f(x)g(x)| < \epsilon$. If $\epsilon > 0$ is given, choose δ_1 so that when $|x-c| < \delta_1$, we have $|f(x)| < \sqrt{\epsilon}$. This can be done since $\lim_{x\to 0} f(x) = 0$. Similarly, choose δ_2 so that when $|x-c| < \delta_2$, we have $|g(x)| < \sqrt{\epsilon}$. Then, if we take δ to be the smaller of δ_1 and δ_2, we'll have that $|x-c| < \delta$ implies both $|f(x)| < \sqrt{\epsilon}$ and $|g(x)| < \sqrt{\epsilon}$. So when $|x-c| < \delta$, we have $|f(x)g(x)| = |f(x)|\,|g(x)| < \sqrt{\epsilon}\cdot\sqrt{\epsilon} = \epsilon$. Thus $\lim_{x\to c} f(x)g(x) = 0$.

(b) $\left(f(x)-L_1\right)\left(g(x)-L_2\right) + L_1g(x) + L_2f(x) - L_1L_2$
$= f(x)g(x) - L_1g(x) - L_2f(x) + L_1L_2 + L_1g(x) + L_2f(x) - L_1L_2 = f(x)g(x).$

(c) $\lim_{x\to c}\left(f(x)-L_1\right) = \lim_{x\to c} f(x) - \lim_{x\to c} L_1 = L_1 - L_1 = 0$, using the second limit property. Similarly, $\lim_{x\to c}\left(g(x)-L_2\right) = 0$.

(d) Since $\lim_{x\to c}\left(f(x)-L_1\right) = \lim_{x\to c}\left(g(x)-L_2\right) = 0$, we have that $\lim_{x\to c}\left(f(x)-L_1\right)\left(g(x)-L_2\right) = 0$ by part (a).

(e) From part (b), we have

$$\begin{aligned}
\lim_{x\to c} f(x)g(x) &= \lim_{x\to c}\left(\left(f(x)-L_1\right)\left(g(x)-L_2\right) + L_1g(x) + L_2f(x) - L_1L_2\right) \\
&= \lim_{x\to c}\left(f(x)-L_1\right)\left(g(x)-L_2\right) + \lim_{x\to c} L_1g(x) + \lim_{x\to c} L_2f(x) + \lim_{x\to c}(-L_1L_2) \\
&\quad \text{(using limit property 2)} \\
&= 0 + L_1\lim_{x\to c} g(x) + L_2\lim_{x\to c} f(x) - L_1L_2 \\
&\quad \text{(using limit property 1 and part (d))} \\
&= L_1L_2 + L_2L_1 - L_1L_2 = L_1L_2.
\end{aligned}$$

Strengthen Your Understanding

34. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \ne 2.99 \\ 1000 & \text{if } x = 2.99. \end{cases}$$

Then $f(2.9) = 2(2.9)+1 = 6.8$, whereas $f(2.99) = 1000$.

35. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \ne 3.01 \\ -1000 & \text{if } x = 3.01. \end{cases}$$

Then $f(3.1) = 2(3.1)+1 = 7.2$, whereas $f(3.01) = -1000$.

36. False. For some functions we need to pick smaller values of δ. For example, if $f(x) = x^{1/3}+2$ and $c = 0$ and $L = 2$, then $f(x)$ is within 10^{-3} of 2 if $|x^{1/3}| < 10^{-3}$. This only happens if x is within $(10^{-3})^3 = 10^{-9}$ of 0. If $x = 10^{-3}$ then $x^{1/3} = (10^{-3})^{1/3} = 10^{-1}$, which is too large.

37. False. The definition of a limit guarantees that, for any positive ϵ, there is a δ. This statement, which guarantees an ϵ for a specific $\delta = 10^{-3}$, is not equivalent to $\lim_{x\to c} f(x) = L$. For example, consider a function with a vertical asymptote within 10^{-3} of 0, such as $c = 0$, $L = 0$, $f(x) = x/(x-10^{-4})$.

38. True. This is equivalent to the definition of a limit.

39. False. Although x may be far from c, the value of $f(x)$ could be close to L. For example, suppose $f(x) = L$, the constant function.

40. False. The definition of the limit says that if x is within δ of c, then $f(x)$ is within ϵ of L, not the other way round.

Solutions for Chapter 1 Review

Exercises

1. The line of slope m through the point (x_0, y_0) has equation

$$y - y_0 = m(x - x_0),$$

so the line we want is

$$\begin{aligned} y - 0 &= 2(x - 5) \\ y &= 2x - 10. \end{aligned}$$

2. We want a function of the form $y = a(x-h)^2 + k$, with $a < 0$ because the parabola opens downward. Since (h, k) is the vertex, we must take $h = 2$, $k = 5$, but we can take any negative value of a. Figure 1.127 shows the graph with $a = -1$, namely $y = -(x-2)^2 + 5$.

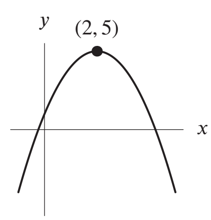


Figure 1.127: Graph of $y = -(x-2)^2 + 5$

3. A parabola with x-intercepts ± 1 has an equation of the form

$$y = k(x-1)(x+1).$$

Substituting the point $x = 0$, $y = 3$ gives

$$3 = k(-1)(1) \quad \text{so} \quad k = -3.$$

Thus, the equation we want is

$$\begin{aligned} y &= -3(x-1)(x+1) \\ y &= -3x^2 + 3. \end{aligned}$$

4. The equation of the whole circle is

$$x^2 + y^2 = (\sqrt{2})^2,$$

so the bottom half is

$$y = -\sqrt{2 - x^2}.$$

5. A circle with center (h, k) and radius r has equation $(x-h)^2+(y-k)^2=r^2$. Thus $h=-1$, $k=2$, and $r=3$, giving

$$(x+1)^2+(y-2)^2=9.$$

Solving for y, and taking the positive square root gives the top half, so

$$\begin{aligned}(y-2)^2 &= 9-(x+1)^2\\ y &= 2+\sqrt{9-(x+1)^2}.\end{aligned}$$

See Figure 1.128.

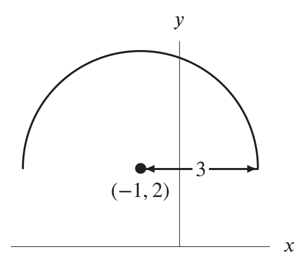


Figure 1.128: Graph of $y=2+\sqrt{9-(x+1)^2}$

6. A cubic polynomial of the form $y=a(x-1)(x-5)(x-7)$ has the correct intercepts for any value of $a \neq 0$. Figure 1.129 shows the graph with $a=1$, namely $y=(x-1)(x-5)(x-7)$.

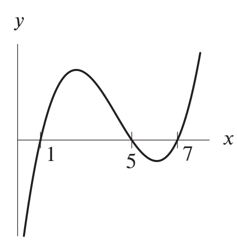


Figure 1.129: Graph of $y=(x-1)(x-5)(x-7)$

7. Since the vertical asymptote is $x=2$, we have $b=-2$. The fact that the horizontal asymptote is $y=-5$ gives $a=-5$. So

$$y=\frac{-5x}{x-2}.$$

8. The amplitude of this function is 5, and its period is 2π, so $y=5\cos x$.

9. See Figure 1.130.

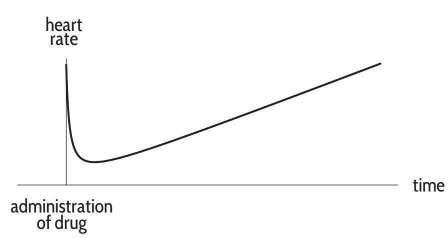


Figure 1.130

10. Factoring gives

$$g(x) = \frac{(2-x)(2+x)}{x(x+1)}.$$

The values of x which make $g(x)$ undefined are $x = 0$ and $x = -1$, when the denominator is 0. So the domain is all $x \neq 0, -1$. Solving $g(x) = 0$ means one of the numerator's factors is 0, so $x = \pm 2$.

11. (a) The domain of f is the set of values of x for which the function is defined. Since the function is defined by the graph and the graph goes from $x = 0$ to $x = 7$, the domain of f is $[0, 7]$.
(b) The range of f is the set of values of y attainable over the domain. Looking at the graph, we can see that y gets as high as 5 and as low as -2, so the range is $[-2, 5]$.
(c) Only at $x = 5$ does $f(x) = 0$. So 5 is the only zero of $f(x)$.
(d) Looking at the graph, we can see that $f(x)$ is decreasing on $(1, 7)$.
(e) The graph indicates that $f(x)$ is concave up at $x = 6$.
(f) The value $f(4)$ is the y-value that corresponds to $x = 4$. From the graph, we can see that $f(4)$ is approximately 1.
(g) This function is not invertible, since it fails the horizontal-line test. A horizontal line at $y = 3$ would cut the graph of $f(x)$ in two places, instead of the required one.

12. (a) $f(n) + g(n) = (3n^2 - 2) + (n + 1) = 3n^2 + n - 1$.
(b) $f(n)g(n) = (3n^2 - 2)(n + 1) = 3n^3 + 3n^2 - 2n - 2$.
(c) The domain of $f(n)/g(n)$ is defined everywhere where $g(n) \neq 0$, i.e. for all $n \neq -1$.
(d) $f(g(n)) = 3(n + 1)^2 - 2 = 3n^2 + 6n + 1$.
(e) $g(f(n)) = (3n^2 - 2) + 1 = 3n^2 - 1$.

13. (a) Since $m = f(A)$, we see that $f(100)$ represents the value of m when $A = 100$. Thus $f(100)$ is the minimum annual gross income needed (in thousands) to take out a 30-year mortgage loan of \$100,000 at an interest rate of 6%.
(b) Since $m = f(A)$, we have $A = f^{-1}(m)$. We see that $f^{-1}(75)$ represents the value of A when $m = 75$, or the size of a mortgage loan that could be obtained on an income of \$75,000.

14. Taking logs of both sides yields $t \log 5 = \log 7$, so $t = \dfrac{\log 7}{\log 5} = 1.209$.

15. $t = \dfrac{\log 2}{\log 1.02} \approx 35.003$.

16. Collecting similar factors yields $\left(\frac{3}{2}\right)^t = \frac{5}{7}$, so

$$t = \frac{\log\left(\frac{5}{7}\right)}{\log\left(\frac{3}{2}\right)} = -0.830.$$

17. Collecting similar factors yields $\left(\frac{1.04}{1.03}\right)^t = \frac{12.01}{5.02}$. Solving for t yields

$$t = \frac{\log\left(\frac{12.01}{5.02}\right)}{\log\left(\frac{1.04}{1.03}\right)} = 90.283.$$

18. We want $2^t = e^{kt}$ so $2 = e^k$ and $k = \ln 2 = 0.693$. Thus $P = P_0 e^{0.693t}$.

19. We want $0.2^t = e^{kt}$ so $0.2 = e^k$ and $k = \ln 0.2 = -1.6094$. Thus $P = 5.23e^{-1.6094t}$.

20. $f(x) = \ln x, \quad g(x) = x^3$. (Another possibility: $f(x) = 3x, \quad g(x) = \ln x$.)

21. $f(x) = x^3, \quad g(x) = \ln x$.

22. The amplitude is 5. The period is 6π. See Figure 1.131.

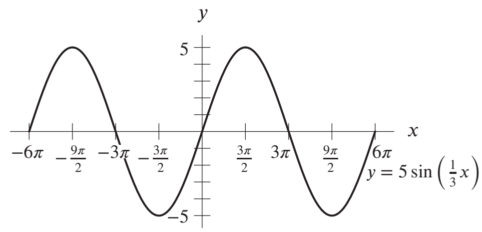


Figure 1.131

23. The amplitude is 2. The period is $2\pi/5$. See Figure 1.132.

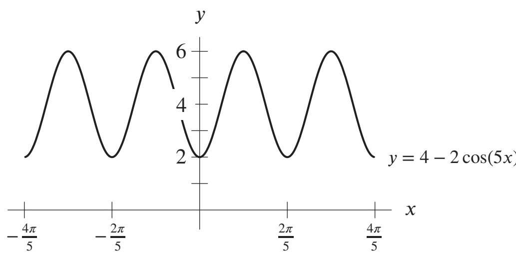
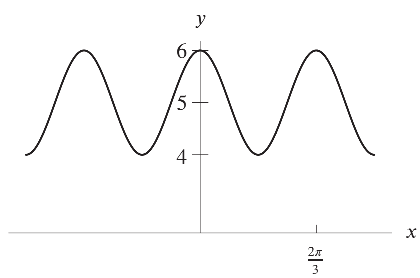


Figure 1.132

24. (a) We determine the amplitude of y by looking at the coefficient of the cosine term. Here, the coefficient is 1, so the amplitude of y is 1. Note that the constant term does not affect the amplitude.

(b) We know that the cosine function $\cos x$ repeats itself at $x = 2\pi$, so the function $\cos(3x)$ must repeat itself when $3x = 2\pi$, or at $x = 2\pi/3$. So the period of y is $2\pi/3$. Here as well the constant term has no effect.

(c) The graph of y is shown in the figure below.



25. (a) Since $f(x)$ is an odd polynomial with a positive leading coefficient, it follows that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

(b) Since $f(x)$ is an even polynomial with negative leading coefficient, it follows that $f(x) \to -\infty$ as $x \to \pm\infty$.

(c) As $x \to \pm\infty$, $x^4 \to \infty$, so $x^{-4} = 1/x^4 \to 0$.

(d) As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in its numerator and denominator. So as $x \to \pm\infty$, $f(x) \to 6$.

26. Exponential growth dominates power growth as $x \to \infty$, so $10 \cdot 2^x$ is larger.

27. As $x \to \infty$, $0.25x^{1/2}$ is larger than $25{,}000x^{-3}$.

28. This is a line with slope $-3/7$ and y-intercept 3, so a possible formula is

$$y = -\frac{3}{7}x + 3.$$

29. Starting with the general exponential equation $y = Ae^{kx}$, we first find that for $(0, 1)$ to be on the graph, we must have $A = 1$. Then to make $(3, 4)$ lie on the graph, we require

$$\begin{aligned} 4 &= e^{3k} \\ \ln 4 &= 3k \\ k &= \frac{\ln 4}{3} \approx 0.4621. \end{aligned}$$

Thus the equation is

$$y = e^{0.4621x}.$$

Alternatively, we can use the form $y = a^x$, in which case we find $y = (1.5874)^x$.

30. This looks like an exponential function. The y-intercept is 3 and we use the form $y = 3e^{kt}$. We substitute the point $(5, 9)$ to solve for k:

$$\begin{aligned} 9 &= 3e^{k5} \\ 3 &= e^{5k} \\ \ln 3 &= 5k \\ k &= 0.2197. \end{aligned}$$

A possible formula is

$$y = 3e^{0.2197t}.$$

Alternatively, we can use the form $y = 3a^t$, in which case we find $y = 3(1.2457)^t$.

31. $y = -kx(x + 5) = -k(x^2 + 5x)$, where $k > 0$ is any constant.

32. Since this function has a y-intercept at $(0, 2)$, we expect it to have the form $y = 2e^{kx}$. Again, we find k by forcing the other point to lie on the graph:

$$\begin{aligned} 1 &= 2e^{2k} \\ \frac{1}{2} &= e^{2k} \\ \ln\left(\frac{1}{2}\right) &= 2k \\ k &= \frac{\ln(\frac{1}{2})}{2} \approx -0.34657. \end{aligned}$$

This value is negative, which makes sense since the graph shows exponential decay. The final equation, then, is

$$y = 2e^{-0.34657x}.$$

Alternatively, we can use the form $y = 2a^x$, in which case we find $y = 2(0.707)^x$.

33. $z = 1 - \cos\theta$

34. $y = k(x + 2)(x + 1)(x - 1) = k(x^3 + 2x^2 - x - 2)$, where $k > 0$ is any constant.

35. $x = ky(y - 4) = k(y^2 - 4y)$, where $k > 0$ is any constant.

36. $y = 5\sin\left(\frac{\pi t}{20}\right)$

37. This looks like a fourth degree polynomial with roots at -5 and -1 and a double root at 3. The leading coefficient is negative, and so a possible formula is

$$y = -(x + 5)(x + 1)(x - 3)^2.$$

38. This looks like a rational function. There are vertical asymptotes at $x = -2$ and $x = 2$ and so one possibility for the denominator is $x^2 - 4$. There is a horizontal asymptote at $y = 3$ and so the numerator might be $3x^2$. In addition, $y(0) = 0$ which is the case with the numerator of $3x^2$. A possible formula is

$$y = \frac{3x^2}{x^2 - 4}.$$

39. There are many solutions for a graph like this one. The simplest is $y = 1 - e^{-x}$, which gives the graph of $y = e^x$, flipped over the x-axis and moved up by 1. The resulting graph passes through the origin and approaches $y = 1$ as an upper bound, the two features of the given graph.

40. The graph is a sine curve which has been shifted up by 2, so $f(x) = (\sin x) + 2$.

41. This graph has period 5, amplitude 1 and no vertical shift or horizontal shift from $\sin x$, so it is given by

$$f(x) = \sin\left(\frac{2\pi}{5}x\right).$$

42. Since the denominator, $x^2 + 1$, is continuous and never zero, $g(x)$ is continuous on $[-1, 1]$.

43. Since

$$h(x) = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)},$$

we see that $h(x)$ is not defined at $x = -1$ or at $x = 1$, so $h(x)$ is not continuous on $[-1, 1]$.

44. (a) $\lim_{x\to 0} f(x) = 1$.
(b) $\lim_{x\to 1} f(x)$ does not exist.
(c) $\lim_{x\to 2} f(x) = 1$.
(d) $\lim_{x\to 3^-} f(x) = 0$.

45. $$f(x) = \frac{x^3|2x-6|}{x-3} = \begin{cases} \dfrac{x^3(2x-6)}{x-3} = 2x^3, & x > 3 \\ \dfrac{x^3(-2x+6)}{x-3} = -2x^3, & x < 3 \end{cases}$$

Figure 1.133 confirms that $\lim_{x\to 3^+} f(x) = 54$ while $\lim_{x\to 3^-} f(x) = -54$; thus $\lim_{x\to 3} f(x)$ does not exist.

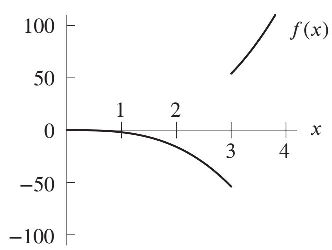


Figure 1.133

46. $$f(x) = \begin{cases} e^x & -1 < x < 0 \\ 1 & x = 0 \\ \cos x & 0 < x < 1 \end{cases}$$

Figure 1.134 confirms that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = e^0 = 1$, and that $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \cos x = \cos 0 = 1$, so $\lim_{x\to 0} f(x) = 1$.

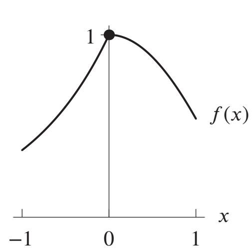


Figure 1.134

47. (a) As t approaches -2 from either side, the values of $g(t)$ get closer and closer to -5, so the limit appears to be about -5.
(b) As t approaches 0 from either side, the values of $g(t)$ get closer and closer to -1, so the limit appears to be about -1.
(c) As t approaches 1, the values of $g(t)$ get closer and closer to 4 on one side of $t = 1$ and get closer and closer to a value less than 1 on the other side of $t = 1$. Thus the limit does not exist.
(d) As t approaches 3 from either side, the values of $g(t)$ become arbitrarily large, so the limit does not exist.

48. (a) As x approaches -2 from either side, the values of $f(x)$ get closer and closer to 3, so the limit appears to be about 3.
(b) As x approaches 0 from either side, the values of $f(x)$ get closer and closer to 7. (Recall that to find a limit, we are interested in what happens to the function near x but not at x.) The limit appears to be about 7.
(c) As x approaches 2 from either side, the values of $f(x)$ get closer and closer to 3 on one side of $x = 2$ and get closer and closer to 2 on the other side of $x = 2$. Thus the limit does not exist.
(d) As x approaches 4 from either side, the values of $f(x)$ get closer and closer to 8. (Again, recall that we don't care what happens right at $x = 4$.) The limit appears to be about 8.

49. Yes, because the denominator is never zero.

50. Yes, because $2x + x^{2/3}$ is defined for all x.

51. No, because $\sin 0 = 0$.

52. No, because $e^x - 1 = 0$ at $x = 0$.

53. For $-1 \leq \theta \leq 1$, $-1 \leq y \leq 1$, the graph of $y = \dfrac{\cos\theta - 1}{\theta}$ is shown in Figure 1.135. The graph suggests that

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0.$$

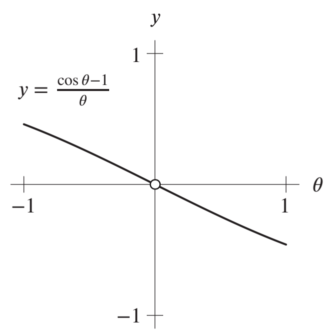


Figure 1.135

54. For $-0.5 \leq \theta \leq 0.5$, $0 \leq y \leq 0.5$, the graph of $y = \dfrac{\theta}{\tan(3\theta)}$ is shown in Figure 1.136. Therefore, by tracing along the curve, we see that $\lim\limits_{\theta\to 0} \dfrac{\theta}{\tan(3\theta)} = 0.3333\ldots$.

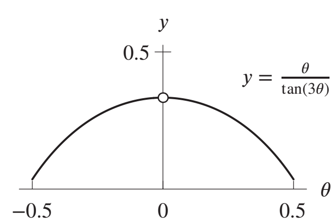


Figure 1.136

55. A graph of $y = \dfrac{3^h - 1}{h}$ in a window such as $-0.5 \leq h \leq 0.5$ and $0 \leq y \leq 1.5$ appears to indicate $y \to 1.1$ as $h \to 0$. By zooming in on the graph, we can estimate the limit more accurately. Therefore, we estimate that

$$\lim_{h\to 0} \frac{3^h - 1}{h} = 1.098.$$

56. A graph of $y = \dfrac{\sin(3h)}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 4$ appears to indicate $y \to 3$ as $h \to 0$. Therefore, we estimate that

$$\lim_{h\to 0} \frac{\sin(3h)}{h} = 3.$$

57. (a) Substituting x-values into $f(x)$ gives:

x	0.9	0.99	1.01	1.1
$f(x)$	1.054	1.005	0.995	0.953

(b) Since the values of $f(x)$ appear to get closer and closer to 1 as x gets closer and closer to 1, we estimate

$$\lim_{x\to 1} \frac{\ln x}{x-1} = 1.$$

Notice that $f(x) = \dfrac{\ln x}{x-1}$ is undefined at $x = 1$.

58. (a) Substituting x-values into $f(x)$ gives:

x	0.9	0.99	1.01	1.1
$f(x)$	1.949	1.995	2.005	2.049

(b) Since the values of $f(x)$ appear to get closer and closer to 2 as x gets closer and closer to 1, we estimate

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 2.$$

Notice that $f(x) = \dfrac{x-1}{\sqrt{x}-1}$ is undefined at $x = 1$.

59. Since the value of x^2 increases faster than the value of $\ln x$, we know that $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2}$ approaches zero as $x \to \infty$.

Thus, $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2} = 0$.

60. Since the value of $e^x - 2$ grows arbitrarily large as $x \to \infty$ while the numerator remains constant, the value of the function approaches zero.

Thus, $\lim\limits_{x\to\infty} \dfrac{2}{e^x - 2} = 0$.

61. For large values of x, $-5x^2 + 2$ behaves like $-5x^2$ and $3x^2$ behaves like $3x^2$, so $(-5x^2+2)/(3x^2)$ behaves like $-5x^2/3x^2$. Thus

$$\lim_{x\to\infty} \frac{-5x^2+2}{3x^2} = \lim_{x\to\infty} \frac{-5x^2}{3x^2} = -\frac{5}{3}.$$

62. Possible answers are $f(x) = x + 4$ and $f(x) = \frac{5x-5}{x-1}$ though answers will vary.

63. Possible answers are $f(x) = \frac{1}{2} + \frac{1}{x}$ and $f(x) = \frac{x^2}{2x^2+3}$ though answers will vary.

64. (a) When $x < 2$, $g(x) = x^2 - x$. So as x approaches 2 from the left, $g(x)$ approaches $2^2 - 2 = 2$. So

$$\lim_{x\to 2^-} g(x) = 2.$$

(b) When $x > 2$, $g(x) = 4/x$. So as x approaches 2 from the right, $g(x)$ approaches $4/2 = 2$. So

$$\lim_{x\to 2^+} g(x) = 2.$$

(c) As x approaches 2 from either side, $g(x)$ approaches 2. Therefore,

$$\lim_{x\to 2} g(x) = 2.$$

(d) For $x > 2$, $g(x) = 4/x$. As x approaches 3 from the left, x is greater than 2, so $g(x)$ approaches $4/3$. Thus

$$\lim_{x\to 3^-} g(x) = \frac{4}{3}.$$

65. (a) For $x < 2$, we have $f(x) = 3x - 5$, so as x approaches 2 from the left, we have

$$\lim_{x\to 2^-} f(x) = 3(2) - 5 = 1.$$

(b) For $x > 2$, we have $f(x) = 3 - 2x$, so as x approaches 2 from the right, we have

$$\lim_{x\to 2^+} f(x) = 3 - 2(2) = -1.$$

(c) The one-sided limits $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ do not agree. Therefore, the limit $\lim_{x\to 2} f(x)$ does not exist, since $f(x)$ does not approach a single value as x approaches 2 from both sides.

(d) If we are looking at the behavior of $f(x)$ close to $x = 0$, then we are looking at values of $f(x)$ for $x \le 2$. So for values of x near zero, $f(x) = 3x - 5$. As x approaches zero, $f(x)$ approaches $3(0) - 5 = -5$. Therefore,

$$\lim_{x\to 0} f(x) = -5.$$

66. (a) When $x < 0$, $p(x) = x + 3$. So as x approaches 0 from the left, $p(x)$ approaches $0 + 3 = 3$. Thus

$$\lim_{x\to 0^-} p(x) = 3.$$

(b) When $x > 0$, $p(x) = x^3 + 1$. So as x approaches 0 from the right, $p(x)$ approaches $0^3 + 1 = 1$. Thus

$$\lim_{x\to 0^+} p(x) = 1.$$

(c) The left-hand and right-hand limits of $p(x)$ at $x = 0$ do not agree. Therefore, $\lim_{x\to 0} p(x)$ does not exist, since there is no single value that $p(x)$ approaches as x approaches 0 from both sides.

(d) No matter what the value of t is, $t^2 \ge 0$. So for all t, $p(t^2) = (t^2)^3 + 1 = t^6 + 1$. So as t approaches 0 from either side, $p(t^2)$ approaches $0^6 + 1 = 1$. Therefore,

$$\lim_{t\to 0} p(t^2) = 1.$$

67. Evaluating $\dfrac{x^2 - 25}{x - 5}$ at $x = 5$ gives us $0/0$, so we see if we can rewrite the function using algebra. We have

$$\frac{x^2 - 25}{x - 5} = \frac{(x + 5)(x - 5)}{x - 5}.$$

Since $x \ne 5$ in the limit, we can cancel the common factor $x - 5$ to see

$$\lim_{x\to 5} \frac{x^2 - 25}{x - 5} = \lim_{x\to 5} \frac{(x + 5)(x - 5)}{x - 5} = \lim_{x\to 5} (x + 5) = 10.$$

68. Evaluating $\dfrac{x - 4}{x^2 - 16}$ at $x = 4$ gives us $0/0$, so we see if we can rewrite the function using algebra. We have

$$\frac{x - 4}{x^2 - 16} = \frac{x - 4}{(x + 4)(x - 4)}.$$

Since $x \ne 4$ in the limit, we can cancel the common factor $x - 4$ to see

$$\lim_{x\to 4} \frac{x - 4}{x^2 - 16} = \lim_{x\to 4} \frac{x - 4}{(x + 4)(x - 4)} = \lim_{x\to 4} \frac{1}{x + 4} = \frac{1}{8}.$$

69. Evaluating $\dfrac{x^2 - 5x + 6}{x - 2}$ at $x = 2$ gives us $0/0$, so we see if we can rewrite the function using algebra. We have

$$\frac{x^2 - 5x + 6}{x - 2} = \frac{(x - 3)(x - 2)}{x - 2}.$$

Since $x \ne 2$ in the limit, we can cancel the common factor $x - 2$ to see

$$\lim_{x\to 2} \frac{x^2 - 5x + 6}{x - 2} = \lim_{x\to 2} \frac{(x - 3)(x - 2)}{x - 2} = \lim_{x\to 2} (x - 3) = -1.$$

70. Evaluating $\dfrac{x-3}{x^2+2x-15}$ at $x=3$ gives us $0/0$, so we see if we can rewrite the function using algebra. We have

$$\frac{x-3}{x^2+2x-15}=\frac{x-3}{(x+5)(x-3)}.$$

Since $x \neq 3$ in the limit, we can cancel the common factor $x-3$ to see

$$\lim_{x\to 3}\frac{x-3}{x^2+2x-15}=\lim_{x\to 3}\frac{x-3}{(x+5)(x-3)}=\lim_{x\to 3}\frac{1}{x+5}=\frac{1}{8}.$$

71. The expression $\dfrac{x^2+8x-2}{x^2-2}$ is continuous everywhere on its domain (that is, at every point except $x=\pm\sqrt{2}$.) In particular, it is continuous at $x=2$ so we find the limit by evaluating the expression at $x=2$. We have

$$\lim_{x\to 2}\frac{x^2+8x-2}{x^2-2}=\frac{18}{2}=9.$$

72. Evaluating the expression at $h=0$, we arrive at $0/0$. Rewriting the expression using algebra gives

$$\begin{aligned}\lim_{h\to 0}\frac{(4+h)^2-4^2}{h}&=\lim_{h\to 0}\frac{(16+8h+h^2)-16}{h}\\&=\lim_{h\to 0}\frac{8h+h^2}{h}\\&=\lim_{h\to 0}\frac{h(8+h)}{h}\\&\text{and since } h\neq 0\text{, canceling the common factor of } h\text{, we have}\\&=\lim_{h\to 0}(8+h)\\&=8.\end{aligned}$$

Problems

73. (a) More fertilizer increases the yield until about 40 lbs.; then it is too much and ruins crops, lowering yield.
(b) The vertical intercept is at $Y=200$. If there is no fertilizer, then the yield is 200 bushels.
(c) The horizontal intercept is at $a=80$. If you use 80 lbs. of fertilizer, then you will grow no apples at all.
(d) The range is the set of values of Y attainable over the domain $0\le a\le 80$. Looking at the graph, we can see that Y goes as high as 550 and as low as 0. So the range is $0\le Y\le 550$.
(e) Looking at the graph, we can see that Y is decreasing at $a=60$.
(f) Looking at the graph, we can see that Y is concave down everywhere, so it is certainly concave down at $a=40$.

74. (a) Given the two points $(0,32)$ and $(100,212)$, and assuming the graph in Figure 1.137 is a line,

$$\text{Slope}=\frac{212-32}{100}=\frac{180}{100}=1.8.$$

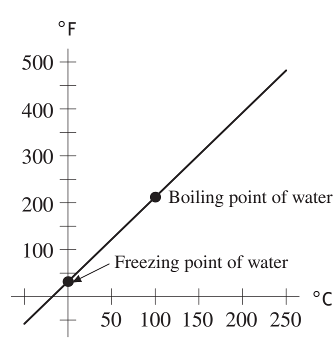


Figure 1.137

(b) The °F-intercept is (0, 32), so

$$°F = 1.8(°C) + 32.$$

(c) If the temperature is 20°Celsius, then

$$°F = 1.8(20) + 32 = 68°F.$$

(d) If °F = °C, then

$$\begin{aligned} °C &= 1.8°C + 32 \\ -32 &= 0.8°C \\ °C &= -40° = °F. \end{aligned}$$

75. (a) We have the following functions.

(i) Since a change in p of \$5 results in a decrease in q of 2, the slope of $q = D(p)$ is $-2/5$ items per dollar. So

$$q = b - \frac{2}{5}p.$$

Now we know that when $p = 550$ we have $q = 100$, so

$$\begin{aligned} 100 &= b - \frac{2}{5} \cdot 550 \\ 100 &= b - 220 \\ b &= 320. \end{aligned}$$

Thus a formula is

$$q = 320 - \frac{2}{5}p.$$

(ii) We can solve $q = 320 - \frac{2}{5}p$ for p in terms of q:

$$\begin{aligned} 5q &= 1600 - 2p \\ 2p &= 1600 - 5q \\ p &= 800 - \frac{5}{2}q. \end{aligned}$$

The slope of this function is $-5/2$ dollars per item, as we would expect.

(b) A graph of $p = 800 - \frac{5}{2}q$ is given in Figure 1.138.

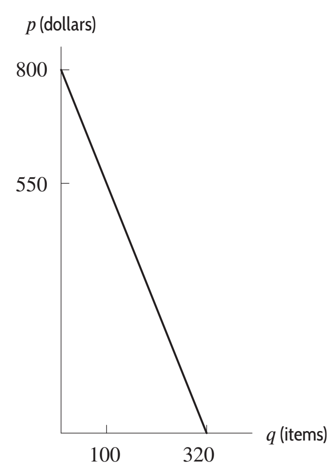


Figure 1.138

76. See Figure 1.139.

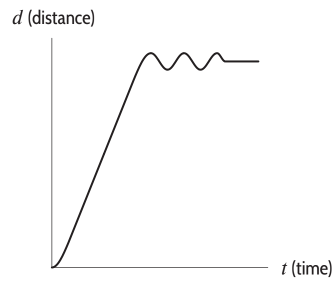


Figure 1.139

77. (a) (i) If the atoms are moved farther apart, then $r > a$ so, from the graph, F is negative, indicating an attractive force, which pulls the atoms back together.

(ii) If the atoms are moved closer together, then $r < a$ so, from the graph, F is positive, indicating an attractive force, which pushes the atoms apart again.

(b) At $r = a$, the force is zero. The answer to part (a)(i) tells us that if the atoms are pulled apart slightly, so $r > a$, the force tends to pull them back together; the answer to part (a)(ii) tells us that if the atoms are pushed together, so $r < a$, the force tends to push them back apart. Thus, $r = a$ is a stable equilibrium.

78. If the pressure at sea level is P_0, the pressure P at altitude h is given by

$$P = P_0\left(1 - \frac{0.4}{100}\right)^{h/100},$$

since we want the pressure to be multiplied by a factor of $(1 - 0.4/100) = 0.996$ for each 100 feet we go up to make it decrease by 0.4% over that interval. At Mexico City $h = 7340$, so the pressure is

$$P = P_0(0.996)^{7340/100} \approx 0.745P_0.$$

So the pressure is reduced from P_0 to approximately $0.745P_0$, a decrease of 25.5%.

79. Assuming the population of Ukraine is declining exponentially, we have population $P(t) = 46.01e^{kt}$ at time t years after 2010. Using the 2014 population, we have

$$\begin{aligned} 45.43 &= 46.01e^{-k\cdot 4} \\ k &= -\frac{1}{4}\ln\left(\frac{45.43}{46.01}\right) = 0.00317. \end{aligned}$$

We want to find the time t at which

$$\begin{aligned} 44 &= 46.01e^{-0.00317t} \\ t &= -\frac{\ln(44/46.01)}{0.00317} = 14.09 \text{ years.} \end{aligned}$$

This model predicts the population to go below 44 million 14.09 years after 2010, in the year 2024.

80. (a) We compound the daily inflation rate 30 times to get the desired monthly rate r:

$$\left(1 + \frac{r}{100}\right)^1 = \left(1 + \frac{0.67}{100}\right)^{30} = 1.2218.$$

Solving for r, we get $r = 22.18$, so the inflation rate for April was 22.18%.

(b) We compound the daily inflation rate 365 times to get a yearly rate R for 2006:

$$\left(1 + \frac{R}{100}\right)^1 = \left(1 + \frac{0.67}{100}\right)^{365} = 11.4426.$$

Solving for R, we get $R = 1044.26$, so the yearly rate was 1044.26% during 2006. We could have obtained approximately the same result by compounding the monthly rate 12 times. Computing the annual rate from the monthly gives a lower result, because 12 months of 30 days each is only 360 days.

81. (a) The US consumption of hydroelectric power increased by at least 20% in 2011 and decreased by at least 10% in 2012, relative to each corresponding previous year. .

(b) False. In 2012 hydroelectric power consumption increased only by 22.1% over consumption in 2011.

(c) True. From 2011 to 2012 consumption decreased by 15.2%, which means $x(1 - 0.152)$ units of hydroelectric power were consumed in 2012 if x had been consumed in 2011. Similarly,

$$(x(1 - 0.152)(1 - 0.025)$$

units of hydroelectric power were consumed in 2013 if x had been consumed in 2012, and

$$(x(1 - 0.152)(1 - 0.025)(1 - 0.036)$$

units of hydroelectric power were consumed in 2014 if x had been consumed in 2013. Since

$$x(1 - 0.152)(1 - 0.025)(1 - 0.036) = x(0.797) = x(1 - 0.203),$$

the percent growth in hydroelectric power consumption was -20.3%, in 2014 relative to consumption in 2011. This amounts to about 20% decrease in hydroelectric power consumption from 2011 to 2014.

82. (a) For each 2.2 pounds of weight the object has, it has 1 kilogram of mass, so the conversion formula is

$$k = f(p) = \frac{1}{2.2}p.$$

(b) The inverse function is

$$p = 2.2k,$$

and it gives the weight of an object in pounds as a function of its mass in kilograms.

83. Since $f(x)$ is a parabola that opens upward, we have $f(x) = ax^2 + bx + c$ with $a > 0$. Since $g(x)$ is a line with negative slope, $g(x) = b + mx$, with slope $m < 0$. Therefore

$$g(f(x)) = b + m(ax^2 + bx + c) = max^2 + mbx + mc + b.$$

The coefficient of x^2 is ma, which is negative. Thus, the graph is a parabola opening downward.

84. (a) is $g(x)$ since it is linear. (b) is $f(x)$ since it has decreasing slope; the slope starts out about 1 and then decreases to about $\frac{1}{10}$. (c) is $h(x)$ since it has increasing slope; the slope starts out about $\frac{1}{10}$ and then increases to about 1.

85. Given the doubling time of 2 hours, $200 = 100e^{k(2)}$, we can solve for the growth rate k using the equation:

$$\begin{aligned} 2P_0 &= P_0e^{2k} \\ \ln 2 &= 2k \\ k &= \frac{\ln 2}{2}. \end{aligned}$$

Using the growth rate, we wish to solve for the time t in the formula

$$P = 100e^{\frac{\ln 2}{2}t}$$

where $P = 3{,}200$, so

$$\begin{aligned} 3{,}200 &= 100e^{\frac{\ln 2}{2}t} \\ t &= 10 \text{ hours}. \end{aligned}$$

86. (a) The y-intercept of $f(x) = a\ln(x+2)$ is $f(0) = a\ln 2$. Since $\ln 2$ is positive, increasing a increases the y-intercept.

(b) The x-intercept of $f(x) = a\ln(x+2)$ is where $f(x) = 0$. Since this occurs where $x + 2 = 1$, so $x = -1$, increasing a does not affect the x-intercept.

87. Since the factor by which the prices have increased after time t is given by $(1.05)^t$, the time after which the prices have doubled solves

$$\begin{aligned} 2 &= (1.05)^t \\ \log 2 &= \log(1.05^t) = t\log(1.05) \\ t &= \frac{\log 2}{\log 1.05} \approx 14.21 \text{ years}. \end{aligned}$$

88. Using the exponential decay equation $P = P_0 e^{-kt}$, we can solve for the substance's decay constant k:

$$\begin{aligned}(P_0 - 0.3P_0) &= P_0 e^{-20k} \\ k &= \frac{\ln(0.7)}{-20}.\end{aligned}$$

Knowing this decay constant, we can solve for the half-life t using the formula

$$\begin{aligned}0.5P_0 &= P_0 e^{\ln(0.7)t/20} \\ t &= \frac{20\ln(0.5)}{\ln(0.7)} \approx 38.87 \text{ hours.}\end{aligned}$$

89. (a) We know the decay follows the equation

$$P = P_0 e^{-kt},$$

and that 10% of the pollution is removed after 5 hours (meaning that 90% is left). Therefore,

$$\begin{aligned}0.90P_0 &= P_0 e^{-5k} \\ k &= -\frac{1}{5}\ln(0.90).\end{aligned}$$

Thus, after 10 hours:

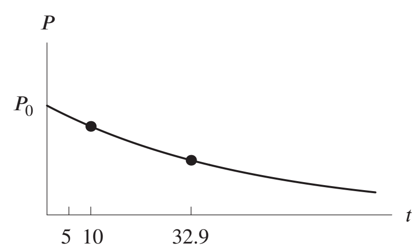
$$\begin{aligned}P &= P_0 e^{-10((-0.2)\ln 0.90)} \\ P &= P_0(0.9)^2 = 0.81P_0\end{aligned}$$

so 81% of the original amount is left.

(b) We want to solve for the time when $P = 0.50P_0$:

$$\begin{aligned}0.50P_0 &= P_0 e^{t((0.2)\ln 0.90)} \\ 0.50 &= e^{\ln(0.90^{0.2t})} \\ 0.50 &= 0.90^{0.2t} \\ t &= \frac{5\ln(0.50)}{\ln(0.90)} \approx 32.9 \text{ hours.}\end{aligned}$$

(c)



(d) When highly polluted air is filtered, there is more pollutant per liter of air to remove. If a fixed amount of air is cleaned every day, there is a higher amount of pollutant removed earlier in the process.

90. Since the amount of strontium-90 remaining halves every 29 years, we can solve for the decay constant;

$$\begin{aligned}0.5P_0 &= P_0 e^{-29k} \\ k &= \frac{\ln(1/2)}{-29}.\end{aligned}$$

Knowing this, we can look for the time t in which $P = 0.10P_0$, or

$$\begin{aligned}0.10P_0 &= P_0 e^{\ln(0.5)t/29} \\ t &= \frac{29\ln(0.10)}{\ln(0.5)} = 96.336 \text{ years.}\end{aligned}$$

91. One hour.

92. (a) V_0 represents the maximum voltage.
(b) The period is $2\pi/(120\pi) = 1/60$ second.
(c) Since each oscillation takes $1/60$ second, in 1 second there are 60 complete oscillations.

93. The US voltage has a maximum value of 156 volts and has a period of $1/60$ of a second, so it executes 60 cycles a second.
The European voltage has a higher maximum of 339 volts, and a slightly longer period of $1/50$ seconds, so it oscillates at 50 cycles per second.

94. (a) The amplitude of the sine curve is $|A|$. Thus, increasing $|A|$ stretches the curve vertically. See Figure 1.140.
(b) The period of the wave is $2\pi/|B|$. Thus, increasing $|B|$ makes the curve oscillate more rapidly—in other words, the function executes one complete oscillation in a smaller interval. See Figure 1.141.

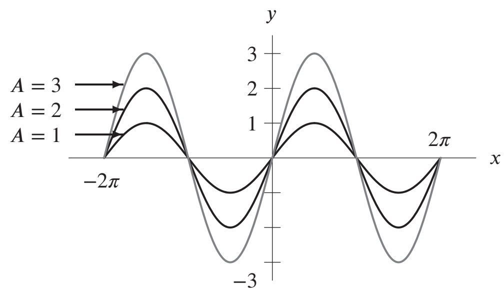


Figure 1.140

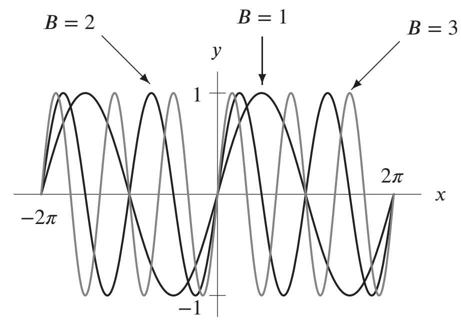


Figure 1.141

95. (a) (i) The water that has flowed out of the pipe in 1 second is a cylinder of radius r and length 3 cm. Its volume is

$$V = \pi r^2(3) = 3\pi r^2.$$

(ii) If the rate of flow is k cm/sec instead of 3 cm/sec, the volume is given by

$$V = \pi r^2(k) = \pi r^2 k.$$

(b) (i) The graph of V as a function of r is a quadratic. See Figure 1.142.

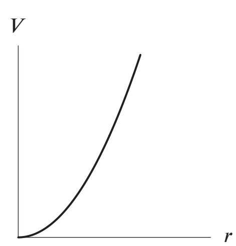


Figure 1.142

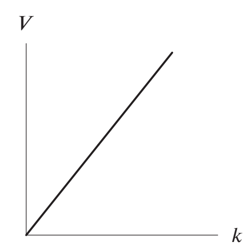


Figure 1.143

(ii) The graph of V as a function of k is a line. See Figure 1.143.

96. Looking at g, we see that the ratio of the values is:

$$\frac{3.12}{3.74} \approx \frac{3.74}{4.49} \approx \frac{4.49}{5.39} \approx \frac{5.39}{6.47} \approx \frac{6.47}{7.76} \approx 0.83.$$

Thus g is an exponential function, and so f and k are the power functions. Each is of the form ax^2 or ax^3, and since $k(1.0) = 9.01$ we see that for k, the constant coefficient is 9.01. Trial and error gives

$$k(x) = 9.01x^2,$$

since $k(2.2) = 43.61 \approx 9.01(4.84) = 9.01(2.2)^2$. Thus $f(x) = ax^3$ and we find a by noting that $f(9) = 7.29 = a(9^3)$ so

$$a = \frac{7.29}{9^3} = 0.01$$

and $f(x) = 0.01x^3$.

97. (a) See Figure 1.144.
(b) The graph is made of straight line segments, rising from the x-axis at the origin to height a at $x = 1$, b at $x = 2$, and c at $x = 3$ and then returning to the x-axis at $x = 4$. See Figure 1.145.

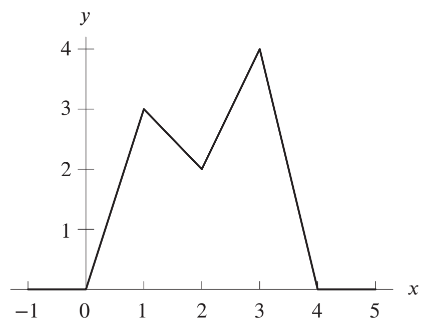


Figure 1.144

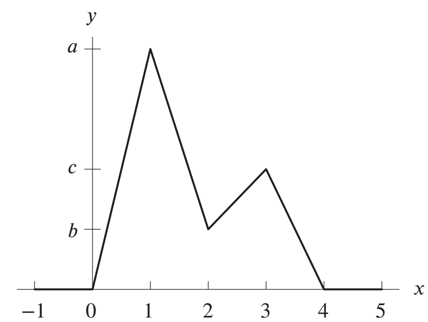


Figure 1.145

98. (a) Reading the graph of θ against t shows that $\theta \approx 5.2$ when $t = 1.5$. Since the coordinates of P are $x = 5\cos\theta$, $y = 5\sin\theta$, when $t = 1.5$ the coordinates are

$$(x, y) \approx (5\cos 5.2, 5\sin 5.2) = (2.3, -4.4).$$

(b) As t increases from 0 to 5, the angle θ increases from 0 to about 6.3 and then decreases to 0 again. Since $6.3 \approx 2\pi$, this means that P starts on the x-axis at the point $(5, 0)$, moves counterclockwise the whole way around the circle (at which time $\theta \approx 2\pi$), and then moves back clockwise to its starting point.

99. (a) III
(b) IV
(c) I
(d) II

100. The functions $y(x) = \sin x$ and $z_k(x) = ke^{-x}$ for $k = 1, 2, 4, 6, 8, 10$ are shown in Figure 1.146. The values of $f(k)$ for $k = 1, 2, 4, 6, 8, 10$ are given in Table 1.9. These values can be obtained using either tracing or a numerical root finder on a calculator or computer.

From Figure 1.146 it is clear that the smallest solution of $\sin x = ke^{-x}$ for $k = 1, 2, 4, 6$ occurs on the first period of the sine curve. For small changes in k, there are correspondingly small changes in the intersection point. For $k = 8$ and $k = 10$, the solution jumps to the second period because $\sin x < 0$ between π and 2π, but ke^{-x} is uniformly positive. Somewhere in the interval $6 \le k \le 8$, $f(k)$ has a discontinuity.

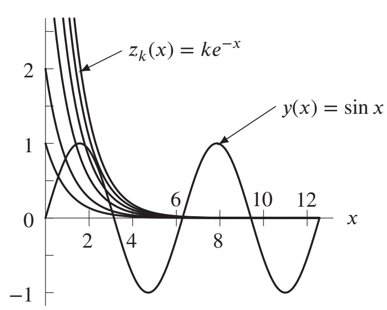


Figure 1.146

Table 1.9

k	$f(k)$
1	0.588
2	0.921
4	1.401
6	1.824
8	6.298
10	6.302

101. For any values of k, the function is continuous on any interval that does not contain $x = 2$.

Since $5x^3 - 10x^2 = 5x^2(x - 2)$, we can cancel $(x - 2)$ provided $x \ne 2$, giving

$$f(x) = \frac{5x^3 - 10x^2}{x - 2} = 5x^2 \qquad x \ne 2.$$

Thus, if we pick $k = 5(2)^2 = 20$, the function is continuous.

102. At $x = 0$, the curve $y = k\cos x$ has $y = k\cos 0 = k$. At $x = 0$, the curve $y = e^x - k$ has $y = e^0 - k = 1 - k$. If $j(x)$ is continuous, we need

$$k = 1 - k, \quad \text{so} \quad k = \frac{1}{2}.$$

103. Two possible graphs are shown in Figures 1.147 and 1.148.

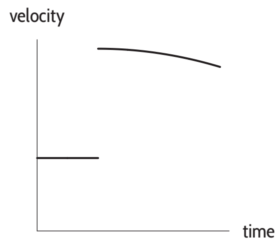


Figure 1.147: Velocity of the car

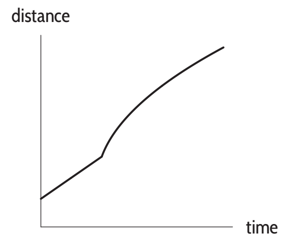


Figure 1.148: Distance

The distance moved by the car is continuous. (Figure 1.148 has no breaks in it.) In actual fact, the velocity of the car is also continuous; however, in this case, it is well-approximated by the function in Figure 1.147, which is not continuous on any interval containing the moment of impact.

104. Figure 1.149 suggests that as $t \to \infty$, $3 - 2e^{-t} \to 3$. We shall use limit properties to confirm this. We now calculate the limit in stages using the properties to justify each step:

$$\begin{aligned}\lim_{t\to\infty}(3 - 2e^{-t}) &= \lim_{t\to\infty}(3) + \lim_{t\to\infty}(-2e^{-t}) && \text{(Property 2)}\\ &= \lim_{t\to\infty}(3) - 2\lim_{t\to\infty}(e^{-t}) && \text{(Property 1)}\\ &= 3 - 2(0) = 3 && \text{(Property 6 and because } e^{-t} \to 0)\end{aligned}$$

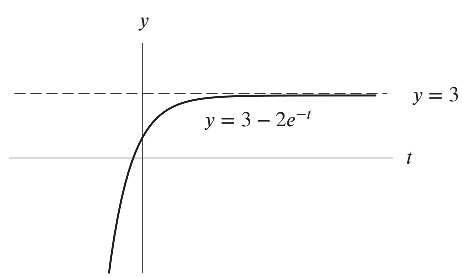


Figure 1.149

105. (a) To find $\lim_{x\to 4^+} f(x)$, we observe how $f(x)$ behaves as x approaches 4 from the right. If $x > 4$, then $f(x) = 9 - x$; so as x approaches 4 from the right, $f(x)$ approaches $9 - 4 = 5$. So $\lim_{x\to 4^+} f(x) = 5$.

(b) When $x < 4$, we have $f(x) = 2x - 3$; so as x approaches 4 from the left, $f(x)$ approaches $2(4) - 3 = 5$. So $\lim_{x\to 4^-} f(x) = 5$.

(c) From (a) and (b), we know $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) = 5$, so $f(x)$ approaches 5 as x approaches 4 from either direction. Therefore, $\lim_{x\to 4} f(x) = 5$.

106. (a) See Figure 1.150.

(b) For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $(x - 2)^2 + 3$ takes on the value $(2 - 2)^2 + 3 = 3$ at $x = 2$, we choose k so that $kx = 3$ at $x = 2$, so $2k = 3$ and $k = 3/2$.

(c) See Figure 1.151.

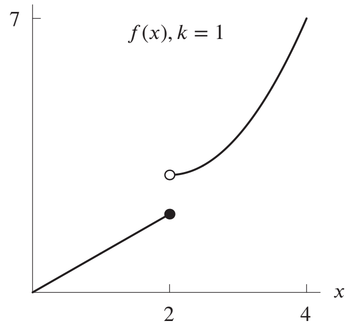


Figure 1.150

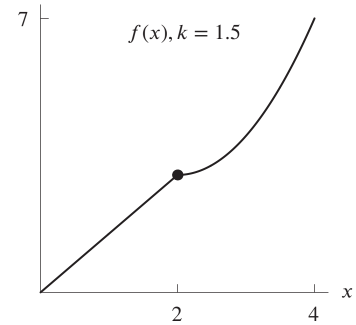


Figure 1.151

107. (a) Figure 1.152 shows a possible graph of $g(x)$, yours may be different.

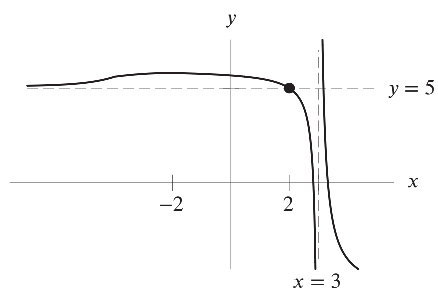


Figure 1.152

(b) In order for g to approach the horizontal asymptote at 5 from above it is necessary that g eventually become concave up. It is therefore not possible for g to be concave down for all $x < -2$.

108. This statement is true; since the graph of $g(t)$ must approach the vertical asymptote at $t = 5$ on the right or the left (or both), $g(t)$ cannot approach a finite real number as t approaches 5 from both sides. So $\lim_{t\to 5} g(t)$ does not exist.

109. This statement is false. Since the graph of $g(t)$ has horizontal asymptotes at both $y = -1$ and at $y = 2$, the graph must approach one of these lines as t approaches ∞ and the other as t approaches $-\infty$. So $\lim_{t\to -\infty} g(t)$ must exist, and must be equal to -1 or 2.

110. We cannot determine whether this statement is true or false. We know that the graph of $g(t)$ has a vertical asymptote at $t = 5$, but we cannot be sure from the information given whether $g(t)$ approaches ∞ or $-\infty$ as t approaches 5. To say that $\lim_{t\to 5} g(t) = \infty$, we would need to know that $g(t)$ approaches ∞ as t approaches 5 from both sides.

111. We cannot determine whether this statement is true or false. We know that the graph of $g(t)$ has horizontal asymptotes at $y = -1$ and at $y = 2$, and the graph must approach each of these. However, it could be that $g(t)$ approaches 2 as t approaches ∞ and -1 as t approaches $-\infty$, or it could be the other way around.

112. (a) The values of $f(x)$ as x gets closer to 5 are getting closer to 3.5, which suggests $\lim_{x\to 5} f(x) = 3.5$.

(b) $f(x)$ is continuous at $x = 5$ if the values of $f(x)$ approach $f(5)$ as x approaches 5. The values approach 3.5, but $f(5) = 8$, so the function is not continuous.

113. (a) The values of $f(x)$ as x gets closer to 5 are getting closer to 3.5, which suggests $\lim_{x\to 5} f(x) = 3.5$.

(b) $f(x)$ is continuous at $x = 5$ if the values of $f(x)$ approach $f(5)$ as x approaches 5. The values approach 3.5, and $f(5) = 3.5$, so the function is continuous.

114. We have,

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} &= \lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}\\ &= \lim_{h\to 0}\frac{x^2-x^2-2xh-h^2}{h(x+h)^2x^2}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{h(-2x-h)}{h(x+h)^2x^2} \\
&= \lim_{h\to 0} \frac{-2x-h}{(x+h)^2x^2} \\
&= \frac{-2x}{x^4} = \frac{-2}{x^3}.
\end{aligned}
$$

115. We have,

$$
\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{x+h)}-\sqrt{x}}{h} &= \lim_{h\to 0} \frac{(\sqrt{x+h)}-\sqrt{x})}{h} \cdot \frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}
$$

116. We have,

$$
\begin{aligned}
\lim_{h\to 0} \frac{2(x+h)-3(x+h)^2-(2x-3x^2)}{h} &= \lim_{h\to 0} \frac{2x+2h-3(x^2+2xh+h^2)-2x+3x^2}{h} \\
&= \lim_{h\to 0} \frac{2x+2h-3x^2-6xh-3h^2-2x+3x^2}{h} \\
&= \lim_{h\to 0} \frac{2h-6xh-3h^2}{h} \\
&= \lim_{h\to 0}(2-6x-3h) \\
&= 2-6x.
\end{aligned}
$$

117.

$$
\begin{aligned}
\lim_{x\to 5} \frac{x^2+4x+3}{x+2} &= \frac{\lim\limits_{x\to 5}(x^2+4x+3)}{\lim\limits_{x\to 5}(x+2)} && \text{Property 4} \\
&= \frac{\lim\limits_{x\to 5}(x^2)+\lim\limits_{x\to 5}(4x)+\lim\limits_{x\to 5}(3)}{\lim\limits_{x\to 5}(x)+\lim\limits_{x\to 5}(2)} && \text{Property 2} \\
&= \frac{(\lim\limits_{x\to 5} x)^2+4\lim\limits_{x\to 5}(x)+\lim\limits_{x\to 5}(3)}{\lim\limits_{x\to 5}(x)+\lim\limits_{x\to 5}(2)} && \text{Properties 1 and 3} \\
&= \frac{5^2+4(5)+3}{5+2} && \text{Properties 5 and 6} \\
&= \frac{48}{7}.
\end{aligned}
$$

118.

$$
\begin{aligned}
\lim_{x\to 0} \frac{2\cos x+12x}{\cos x-10} &= \frac{\lim\limits_{x\to 0}(2\cos x+12x)}{\lim\limits_{x\to 0}(\cos x-10)} && \text{Property 4} \\
&= \frac{\lim\limits_{x\to 0}(2\cos x)+\lim\limits_{x\to 0}(12x)}{\lim\limits_{x\to 0}(\cos x)-\lim\limits_{x\to 0}(10)} && \text{Property 2} \\
&= \frac{2\lim\limits_{x\to 0}(\cos x)+12\lim\limits_{x\to 0}(x)}{\lim\limits_{x\to 0}(\cos x)-\lim\limits_{x\to 0}(10)} && \text{Property 1}
\end{aligned}
$$

$$= \frac{2(1) + 12(0)}{1 - 10} \qquad \text{Properties 5 and 6}$$
$$= -\frac{2}{9}.$$

119. We cannot use the limit properties to calculate this limit since $\lim_{x\to 9} \sqrt{x} - 3 = 0$. However, the limit may be obtained by another method.

120.

$$\begin{aligned}
\lim_{x\to -2} \frac{x^2+5x}{x+9} &= \frac{\lim_{x\to -2}(x^2+5x)}{\lim_{x\to -2}(x+9)} && \text{Property 4}\\
&= \frac{\lim_{x\to -2}(x^2) + \lim_{x\to -2}(5x)}{\lim_{x\to -2}(x) + \lim_{x\to -2}(9)} && \text{Property 2}\\
&= \frac{\lim_{x\to -2}(x^2) + 5\lim_{x\to -2}(x)}{\lim_{x\to -2}(x) + \lim_{x\to -2}(9)} && \text{Properties 1 and 3}\\
&= \frac{(-2)^2 + 5(-2)}{-2+9} && \text{Properties 5 and 6}\\
&= -\frac{6}{7}.
\end{aligned}$$

121. We have

$$\lim_{x\to 4} 2x + 3 = \left(\lim_{x\to 4} 2\right)\left(\lim_{x\to 4} x\right) + \left(\lim_{x\to 4} 3\right) = 2\cdot 4 + 3 = 11.$$

122. We have

$$\lim_{x\to 3} \frac{1}{x} = \frac{\lim_{x\to 3} 1}{\lim_{x\to 3} x} = \frac{1}{3}.$$

123. We have

$$\lim_{x\to 5} x^2 = \lim_{x\to 5} x\cdot x = \left(\lim_{x\to 5} x\right)\left(\lim_{x\to 5} x\right) = 5\cdot 5 = 25.$$

124. We have

$$\lim_{x\to 0} 3x - 5 = \left(\lim_{x\to 0} 3\right)\left(\lim_{x\to 0} x\right) + \left(\lim_{x\to 0} -5\right) = 3\cdot 0 - 5 = -5,$$

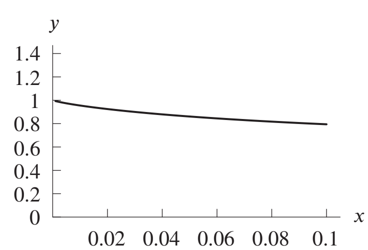
and

$$\lim_{x\to 0} x^2 + 4 = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} x\right) + \left(\lim_{x\to 0} 4\right) = 0\cdot 0 + 4 = 4.$$

Thus,

$$\lim_{x\to 0} \frac{3x-5}{x^2+4} = \frac{\lim_{x\to 0} 3x - 5}{\lim_{x\to 0} x^2 + 4} = \frac{-5}{4}.$$

125. The limit appears to be 1; a graph and table of values is shown below.



x	x^x
0.1	0.7943
0.01	0.9550
0.001	0.9931
0.0001	0.9990
0.00001	0.9999

126. There are many possible correct answers.

(a) If $f(x) = 2x$ and $g(x) = -x$ then $f(x) + g(x) = x$ so $\lim_{x\to\infty} f(x) + g(x) = \infty$.
(b) If $f(x) = x$ and $g(x) = -x + 3$ then $f(x) + g(x) = 3$ so $\lim_{x\to\infty} f(x) + g(x) = 3$.
(c) If $f(x) = x$ and $g(x) = -2x$ then $f(x) + g(x) = -x$ so $\lim_{x\to\infty} f(x) + g(x) = -\infty$.

127. (a) Since $x \cdot \frac{1}{x} = 1$ for all $x \neq 0$, we must have

$$\lim_{x\to 0}\left(x \cdot \frac{1}{x}\right) = 1.$$

(b) Property 3 of the limit properties states

$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$$

provided the limits on the right hand side exist. In this case, $\lim_{x\to 0}\frac{1}{x}$ does not exist, so we cannot invoke the property.

128. From Table 1.10, it appears the limit is 1. This is confirmed by Figure 1.153. An appropriate window is $-0.0033 < x < 0.0033$, $0.99 < y < 1.01$.

Table 1.10

x	$f(x)$
0.1	1.3
0.01	1.03
0.001	1.003
0.0001	1.0003

x	$f(x)$
−0.0001	0.9997
−0.001	0.997
−0.01	0.97
−0.1	0.7

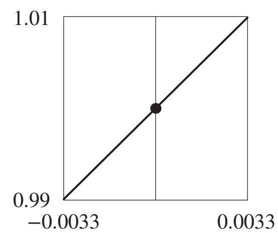


Figure 1.153

129. From Table 1.11, it appears the limit is −1. This is confirmed by Figure 1.154. An appropriate window is $-0.099 < x < 0.099$, $-1.01 < y < -0.99$.

Table 1.11

x	$f(x)$
0.1	−0.99
0.01	−0.9999
0.001	−0.999999
0.0001	−0.99999999

x	$f(x)$
−0.0001	−0.99999999
−0.001	−0.999999
−0.01	−0.9999
−0.1	−0.99

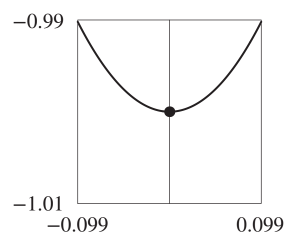


Figure 1.154

130. From Table 1.12, it appears the limit is 0. This is confirmed by Figure 1.155. An appropriate window is $-0.005 < x < 0.005$, $-0.01 < y < 0.01$.

Table 1.12

x	$f(x)$
0.1	0.1987
0.01	0.0200
0.001	0.0020
0.0001	0.0002

x	$f(x)$
−0.0001	−0.0002
−0.001	−0.0020
−0.01	−0.0200
−0.1	−0.1987

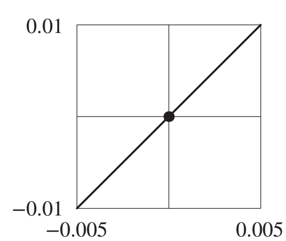


Figure 1.155

131. From Table 1.13, it appears the limit is 0. This is confirmed by Figure 1.156. An appropriate window is $-0.0033 < x < 0.0033$, $-0.01 < y < 0.01$.

Table 1.13

x	$f(x)$
0.1	0.2955
0.01	0.0300
0.001	0.0030
0.0001	0.0003

x	$f(x)$
−0.0001	−0.0003
−0.001	−0.0030
−0.01	−0.0300
−0.1	−0.2955

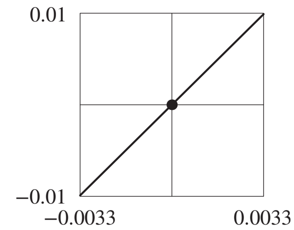


Figure 1.156

132. From Table 1.14, it appears the limit is 2. This is confirmed by Figure 1.157. An appropriate window is $-0.0865 < x < 0.0865$, $1.99 < y < 2.01$.

Table 1.14

x	$f(x)$
0.1	1.9867
0.01	1.9999
0.001	2.0000
0.0001	2.0000

x	$f(x)$
−0.0001	2.0000
−0.001	2.0000
−0.01	1.9999
−0.1	1.9867

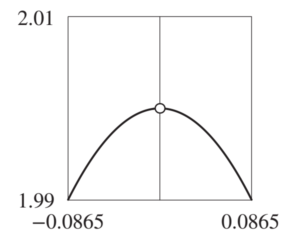


Figure 1.157

133. From Table 1.15, it appears the limit is 3. This is confirmed by Figure 1.158. An appropriate window is $-0.047 < x < 0.047$, $2.99 < y < 3.01$.

Table 1.15

x	$f(x)$
0.1	2.9552
0.01	2.9996
0.001	3.0000
0.0001	3.0000

x	$f(x)$
−0.0001	3.0000
−0.001	3.0000
−0.01	2.9996
−0.1	2.9552

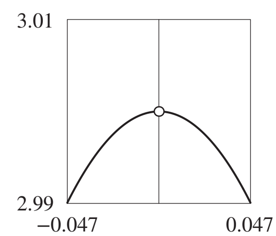


Figure 1.158

134. From Table 1.16, it appears the limit is 1. This is confirmed by Figure 1.159. An appropriate window is $-0.0198 < x < 0.0198$, $0.99 < y < 1.01$.

Table 1.16

x	$f(x)$
0.1	1.0517
0.01	1.0050
0.001	1.0005
0.0001	1.0001

x	$f(x)$
−0.0001	1.0000
−0.001	0.9995
−0.01	0.9950
−0.1	0.9516

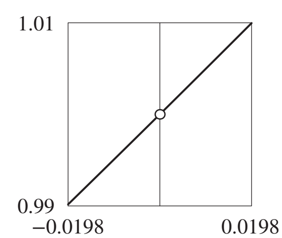


Figure 1.159

135. From Table 1.17, it appears the limit is 2. This is confirmed by Figure 1.160. An appropriate window is $-0.0049 < x < 0.0049$, $1.99 < y < 2.01$.

Table 1.17

x	$f(x)$
0.1	2.2140
0.01	2.0201
0.001	2.0020
0.0001	2.0002

x	$f(x)$
−0.0001	1.9998
−0.001	1.9980
−0.01	1.9801
−0.1	1.8127

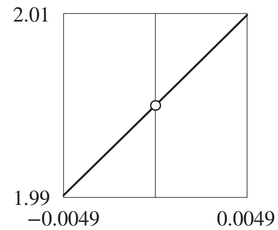


Figure 1.160

CAS Challenge Problems

136. (a) A CAS gives $f(x) = (x - a)(x + a)(x + b)(x - c)$.

(b) The graph of $f(x)$ crosses the x-axis at $x = a$, $x = -a$, $x = -b$, $x = c$; it crosses the y-axis at a^2bc. Since the coefficient of x^4 (namely 1) is positive, the graph of f looks like that shown in Figure 1.161.

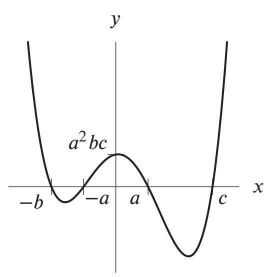


Figure 1.161: Graph of $f(x) = (x - a)(x + a)(x + b)(x - c)$

137. (a) A CAS gives $f(x) = -(x - 1)^2(x - 3)^3$.

(b) For large $|x|$, the graph of $f(x)$ looks like the graph of $y = -x^5$, so $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. The answer to part (a) shows that f has a double root at $x = 1$, so near $x = 1$, the graph of f looks like a parabola touching the x-axis at $x = 1$. Similarly, f has a triple root at $x = 3$. Near $x = 3$, the graph of f looks like the graph of $y = x^3$, flipped over the x-axis and shifted to the right by 3, so that the "seat" is at $x = 3$. See Figure 1.162.

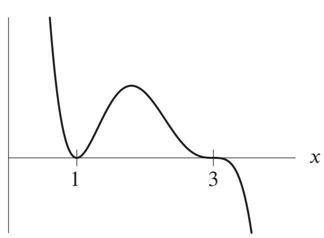


Figure 1.162: Graph of $f(x) = -(x - 1)^2(x - 3)^3$

138. (a) As $x \to \infty$, the term e^{6x} dominates and tends to ∞. Thus, $f(x) \to \infty$ as $x \to \infty$.
As $x \to -\infty$, the terms of the form e^{kx}, where $k = 6, 5, 4, 3, 2, 1$, all tend to zero. Thus, $f(x) \to 16$ as $x \to -\infty$.

(b) A CAS gives

$$f(x) = (e^x + 1)(e^{2x} - 2)(e^x - 2)(e^{2x} + 2e^x + 4).$$

Since e^x is always positive, the factors $(e^x + 1)$ and $(e^{2x} + 2e^x + 4)$ are never zero. The other factors each lead to a zero, so there are two zeros.

(c) The zeros are given by

$$e^{2x} = 2 \quad \text{so} \quad x = \frac{\ln 2}{2}$$
$$e^x = 2 \quad \text{so} \quad x = \ln 2.$$

Thus, one zero is twice the size of the other.

139. (a) Since $f(x) = x^2 - x$,

$$f(f(x)) = (f(x))^2 - f(x) = (x^2 - x)^2 - (x^2 - x) = x - 2x^3 + x^4.$$

Using the CAS to define the function $f(x)$, and then asking it to expand $f(f(f(x)))$, we get

$$f(f(f(x))) = -x + x^2 + 2x^3 - 5x^4 + 2x^5 + 4x^6 - 4x^7 + x^8.$$

(b) The degree of $f(f(x))$ (that is, f composed with itself 2 times) is $4 = 2^2$. The degree of $f(f(f(x)))$ (that is, f composed with itself 3 times), is $8 = 2^3$. Each time you substitute f into itself, the degree is multiplied by 2, because you are substituting in a degree 2 polynomial. So we expect the degree of $f(f(f(f(f(f(x))))))$ (that is, f composed with itself 6 times) to be $64 = 2^6$.

140. (a) A CAS or division gives

$$f(x) = \frac{x^3 - 30}{x - 3} = x^2 + 3x + 9 - \frac{3}{x - 3},$$

so $p(x) = x^2 + 3x + 9$, and $r(x) = -3$, and $q(x) = x - 3$.

(b) The vertical asymptote is $x = 3$. Near $x = 3$, the values of $p(x)$ are much smaller than the values of $r(x)/q(x)$. Thus

$$f(x) \approx \frac{-3}{x - 3} \qquad \text{for } x \text{ near } 3.$$

(c) For large x, the values of $p(x)$ are much larger than the value of $r(x)/q(x)$. Thus

$$f(x) \approx x^2 + 3x + 9 \qquad \text{as } x \to \infty,\ x \to -\infty.$$

(d) Figure 1.163 shows $f(x)$ and $y = -3/(x - 3)$ for x near 3. Figure 1.164 shows $f(x)$ and $y = x^2 + 3x + 9$ for $-20 \le x \le 20$. Note that in each case the graphs of f and the approximating function are close.

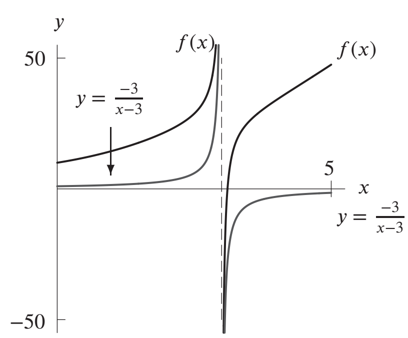


Figure 1.163: Close-up view of $f(x)$ and $y = -3/(x - 3)$

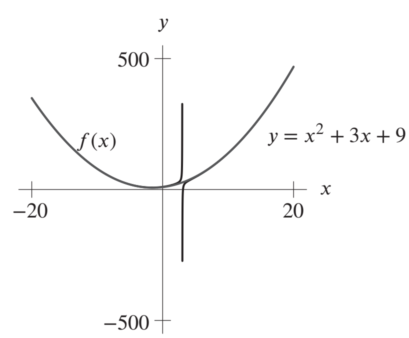


Figure 1.164: Far-away view of $f(x)$ and $y = x^2 + 3x + 9$

141. Using the trigonometric expansion capabilities of your CAS, you get something like

$$\sin(5x) = 5\cos^4(x)\,\sin(x) - 10\cos^2(x)\sin^3(x) + \sin^5(x).$$

Answers may vary. To get rid of the powers of cosine, use the identity $\cos^2(x) = 1 - \sin^2(x)$. This gives

$$\sin(5x) = 5\ \sin(x)\left(1 - \sin^2(x)\right)^2 - 10\sin^3(x)\ \left(1 - \sin^2(x)\right) + \sin^5(x).$$

Finally, using the CAS to simplify,

$$\sin(5x) = 5\ \sin(x) - 20\sin^3(x) + 16\sin^5(x).$$

142. Using the trigonometric expansion capabilities of your computer algebra system, you get something like

$$\cos(4x) = \cos^4(x) - 6\cos^2(x)\sin^2(x) + \sin^4(x).$$

Answers may vary.

(a) To get rid of the powers of cosine, use the identity $\cos^2(x) = 1 - \sin^2(x)$. This gives

$$\cos(4x) = \cos^4(x) - 6\cos^2(x)\,\left(1 - \cos^2(x)\right) + \left(1 - \cos^2(x)\right)^2.$$

Finally, using the CAS to simplify,

$$\cos(4x) = 1 - 8\cos^2(x) + 8\cos^4(x).$$

(b) This time we use $\sin^2(x) = 1 - \cos^2(x)$ to get rid of powers of sine. We get

$$\cos(4x) = \left(1 - \sin^2(x)\right)^2 - 6\sin^2(x)\,\left(1 - \sin^2(x)\right) + \sin^4(x) = 1 - 8\sin^2(x) + 8\sin^4(x).$$

PROJECTS FOR CHAPTER ONE

1. Notice that whenever x increases by 0.5, $f(x)$ increases by 1, indicating that $f(x)$ is linear. By inspection, we see that $f(x) = 2x$.

Similarly, $g(x)$ decreases by 1 each time x increases by 0.5. We know, therefore, that $g(x)$ is a linear function with slope $\frac{-1}{0.5} = -2$. The y-intercept is 10, so $g(x) = 10 - 2x$.

$h(x)$ is an even function which is always positive. Comparing the values of x and $h(x)$, it appears that $h(x) = x^2$.

$F(x)$ is an odd function that seems to vary between -1 and 1. We guess that $F(x) = \sin x$ and check with a calculator.

$G(x)$ is also an odd function that varies between -1 and 1. Notice that $G(x) = F(2x)$, and thus $G(x) = \sin 2x$.

Notice also that $H(x)$ is exactly 2 more than $F(x)$ for all x, so $H(x) = 2 + \sin x$.

2. (a) Begin by finding a table of correspondences between the mathematicians' and meteorologists' angles.

θ_{met} (in degrees)	0	45	90	135	180	225	270	315
θ_{math} (in degrees)	270	225	180	135	90	45	0	315

The table is linear for $0 \leq \theta_{met} \leq 270$, with θ_{math} decreasing by 45 every time θ_{met} increases by 45, giving slope $\Delta\theta_{met}/\Delta\theta_{math} = 45/(-45) = -1$.

The interval $270 < \theta_{met} < 360$ needs a closer look. We have the following more detailed table for that interval:

θ_{met}	280	290	300	310	320	330	340	350
θ_{math}	350	340	330	320	310	300	290	280

Again the table is linear, this time with θ_{math} decreasing by 10 every time θ_{met} increases by 10, again giving slope -1. The graph of θ_{math} against θ_{met} contains two straight line sections, both of slope -1. See Figure 1.165.

(b) See Figure 1.165.

$$\theta_{math} = \begin{cases} 270 - \theta_{met} & \text{if } 0 \leq \theta_{met} \leq 270 \\ 630 - \theta_{met} & \text{if } 270 < \theta_{met} < 360 \end{cases}.$$

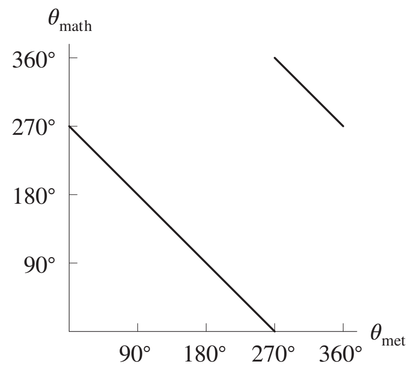


Figure 1.165